St John's College

UPPER V

Advanced Programme Mathematics

July 2011 **Time: 3 hours** **Marks: 300**

Examiner: Mr G Evans
Moderator: Mrs K Jacobs

INSTRUCTIONS AND INFORMATION

Read the following instructions carefully

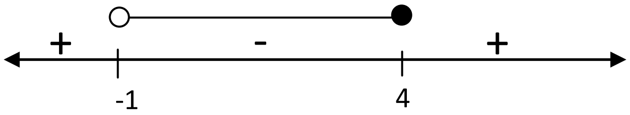
1. This paper consists of 9 pages, including an information sheet. Please make sure your paper is complete.
2. Read the questions carefully.
3. Plan your time carefully – approx 2 hours Section A and 1 hour Section B.
4. Answer all the questions.
5. Number your answers exactly as the questions are numbered.
6. You may use an approved non-programmable and non-graphical calculator, unless otherwise stated.
7. Round off your answers to one decimal digit, where appropriate.
8. All the necessary and reasonable working details must be clearly shown.
9. It is in your own interest to write legibly and to present your work neatly.

Section A – Calculus and Algebra (200 marks)

Question 1

(a) Solve the inequality: $\frac{5x}{x+1} \leq 4$ (6)

$$\frac{5x}{x+1} - 4 \leq 0$$
$$\frac{x-4}{x+1} \leq 0$$
$$-1 < x \leq 4$$



(b) Angie solves a quadratic equation using the formula, and gets the result:

$$x = \frac{-2 \pm \sqrt{m-8k}}{2} \quad \Longrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(1) Write down the value of m. (1)

$$m = b^2 = 2^2 = 4$$

(2) What is the coefficient of x^2 in the original equation? (2)

From denominator 2a = 2. Therefore a = 1

(3) For which values of k will the equation produce solutions with an imaginary part. (3)

$$4 - 8k < 0$$
$$\therefore k > \tfrac{1}{2}$$

(c) If $(a+3i)(4-i) = (11+bi)$, find the values of a and b respectively. (8)

$$(a+3i)(4-i) = (11+bi)$$
$$(4a+3) + i(12-a) = 11 + bi$$
$$\therefore 4a + 3 = 11 \quad and \quad 12 - a = b$$
$$a = 2$$
$$b = 10$$

(d) It is given that:

$$|\ln x| - 2 = \frac{p}{|\ln x|}$$

If $x = e^3$ is one solution, then find the value of p and hence any other solutions. (10)

[30]

$$\left|\ln e^3\right| - 2 = \frac{p}{\left|\ln e^3\right|}$$

$$\therefore 3 - 2 = \frac{p}{3}$$

$$\therefore p = 3$$

$$let\ \ k = \left|\ln x\right|$$

$$k - 2 = \frac{3}{k}$$

$$k^2 - 2k - 3 = 0$$

$$k = 3,\ k = -1$$

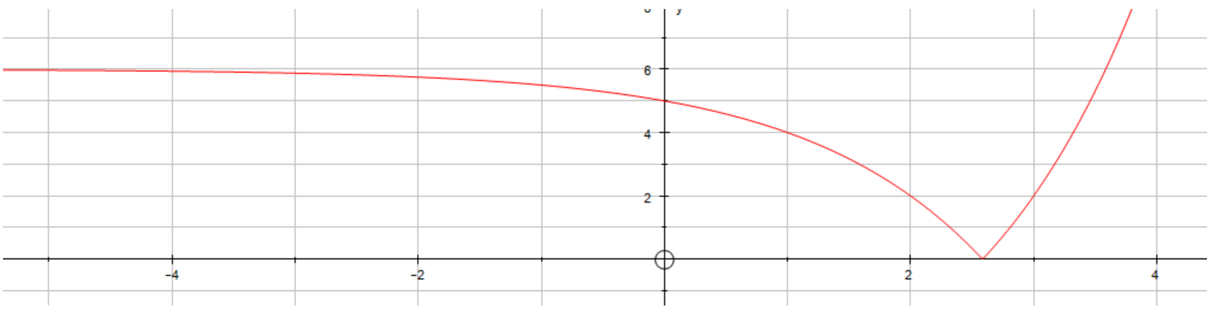
$$|\ln x| = 3 \qquad |\ln x| \neq -1$$

$$\ln x = \pm 3$$

$$x = e^3 \ \ o \quad x = e^{-3}$$

Question 2

(a) Sketch the graph: $f(x) = \left|2^x - 6\right|$ (6)



(b) Hence, or otherwise, solve the inequality: $\left|2^x - 6\right| < 2$ (6)

$$\left|2^x - 6\right| = 2$$

$2^x = 8 \qquad\qquad -2^x + 6 = 2$

$x = 3 \qquad\qquad 2^x = 4 \therefore x = 2$

$\therefore$ Looking at graph: $2 < x < 3$

[12]

Question 3

Prove, by induction, that $5^{n+1}+2^{n}$ is a multiple of 3 for all $n \in N$ **[12]**

(1) If n = 1:

$5^2+2=27$ which is a multiple of 3

(2) Assume initially that: $5^{k+1}+2^{k}=3m \quad m \in N$

(3) Now prove rule holds for n = k + 1:

$5^{k+2}+2^{k+1}$

$=25.5^k+2.2^k$

$=25.5^k+2\left(3m-5^{k+1}\right)$

$=25.5^k+6m-10.5^k$

$=15.5^k+6m$

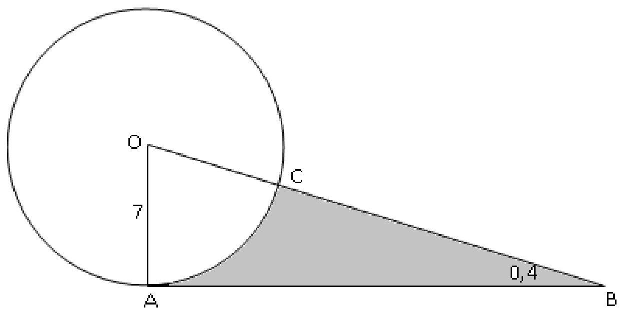
$=3\left(5^{k+1}+2m\right)$

which is clearly divisible by 3.

(4) In summary, we have shown that if the rule is true for n = k, then it is certainly true for n = k + 1. By this we know that the rule is true for n = 2, since we showed it was true for n = 1. By the same argument, it then true for n = 3, n = 4, n = 5 and all natural values of n.

Question 4

AB is a tangent to the circle centre O.
$A\hat{B}O=0,4$ radians. OA = 7 cm.



(a) Write down the size of $A\hat{O}C$ (2)

$\frac{\pi}{2}-0,4$

$=1,17 \ rad$

(b) Find the area of the shaded region. (9)

$\tan 0,4=\frac{7}{AB}$

$\therefore AB=16,56$

$Area \ \Delta OAB=\frac{7\times 16,56}{2}=57,95 \ units^2$

$Area \, OAC=\frac{1}{2}(7)^2\times 1,17=28,68 \ units^2$

$\therefore shaded \ region=57,95-28,68=29,26 \, units^2$

[11]

Question 5

Determine the limits:

(a)

(8)

$$\lim_{x\to 6}\frac{\sqrt{4x+1}-\sqrt{3x+7}}{x-6}$$
$$=\lim_{x\to 6}\frac{(4x+1)-(3x+7)}{(x-6)\left(\sqrt{4x+1}+\sqrt{3x+7}\right)}$$
$$=\lim_{x\to 6}\frac{x-6}{(x-6)\left(\sqrt{4x+1}+\sqrt{3x+7}\right)}$$
$$=\frac{1}{\sqrt{25}+\sqrt{25}}$$
$$=\frac{1}{10}$$

(b)

$$\lim_{x\to 0}\frac{\sin 4x\cos x+\sin x\cos 4x}{x}$$
$$=\lim_{x\to 0}\frac{\sin 5x}{x}$$

(6)

$$=5\times\lim\frac{\sin 5x}{5x}$$
$$=5\times 1=5$$

[14]

Question 6

(a) Consider the function f defined below:

$$f(x)=\begin{cases} 2^x+a & if \quad x<3 \\ 2b-3a & if \quad x=3 \\ \frac{b}{x-2} & if \quad x>3 \end{cases}$$

Given that the function is continuous at $x = 3$, determine the values of a and b respectively. (11)

$$\lim_{x\to 3^-} f(x) = 2^3 + a = 8 + a$$
$$\lim_{x\to 3^+} f(x) = \tfrac{b}{3-2} = b$$
$$\therefore b = 8 + a$$
$$f(3) = 2b - 3a$$
$$\therefore 2b - 3a = b$$
$$\therefore b = 3a$$
$$\therefore 3a = 8 + a$$
$$a = 4, \quad b = 12$$

(b) Consider the following functions and write in each case:
(i) the type of discontinuity
(ii) the equation of any asymptotes

$f(x) = \dfrac{x^2 - 9}{x - 3}$

$= x + 3$

Removable disc at x = 3

$g(x) = \dfrac{x^2 - 9}{x^2 - 4}$

vert. asympt. x = ± 2

jump disc: x = ± 2

horiz. asympt. y = 1

$h(x) = \dfrac{x^2 + 1}{x}$

vert. asympt. x = 0

jump disc. x = 0

oblique asympt. y = x

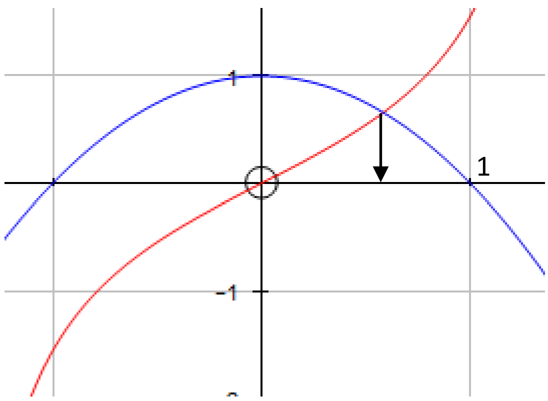
(16)

[27]

Question 7

It is required to solve: $\tan x = 1 - x^2$ using Newton's method.

(a) By drawing a suitable rough sketch, show that a reasonable first approximation is $x_1 = 0{,}7$ (4)



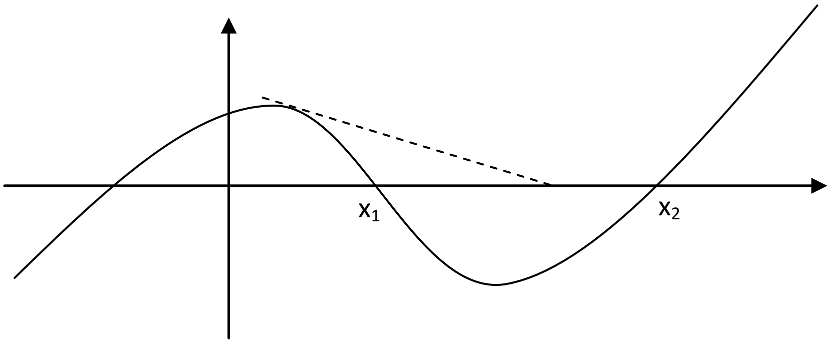
(b) Hence solve for the nearest solution, correct to 5 decimal places (8)

$$\tan x = 1 - x^2$$
$$f(x) = \tan x + x^2 - 1$$
$$f'(x) = \sec^2 x + 2x$$
$$x_{r+1} = x_r - \frac{\tan x + x^2 - 1}{\frac{1}{\cos^2 x} + 2x}$$
$$x_1 = 0,7$$
$$x_2 = 0.593135953$$
$$x_3 = 0.58332137$$
$$x_4 = 0.583248471$$
$$x_5 = 0.583248467$$

Therefore x = 0,58325 (5 dp)

(c) Does the initial guess in Newton's method need to be close to the actual answer?
With the aid of a sketch, explain why and/or why not? (5)

The initial guess needs to be close if there are other solutions nearby. Note the tangent near x1 can converge to the solution x2 since it is quite close to the turning point. However, any guess to the right of x2 will converge to x2 so in this case the initial guess does not need to be close.



[17]

Question 8

(a) Find $\frac{dy}{dx}$ if $y = \sqrt{2x+1}.\tan\left(\sqrt{2x+1}\right)$ Simplify your answer. (10)

$$\frac{dy}{dx} = \tan\left(\sqrt{2x+1}\right).\frac{1}{2}(2x+1)^{-\frac{1}{2}}. \quad + 2\sqrt{2x+1}.\sec^2\left(\sqrt{2x+1}\right).\frac{1}{2}(2x+1)^{-\frac{1}{2}}.$$
$$= \frac{\tan\left(\sqrt{2x+1}\right)}{\sqrt{2x+1}} + \sec^2\left(\sqrt{2x+1}\right)$$

(b) Determine the equation of the tangent to the curve $y^2 + xy = 8$ at the point (-2; 4). (11)

$$y^2 + xy = 8$$
$$2y.\frac{dy}{dx} + y.1 + x.\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(2y + x) = -y$$
$$\frac{dy}{dx} = \frac{-y}{2y + x}$$
$$\frac{dy}{dx} = \frac{-4}{2(-4) - 2} = \frac{2}{5}$$
$$y - 4 = \frac{2}{5}(x + 2)$$
$$y = \tfrac{2}{5}x + \tfrac{24}{5}$$

[21]

Question 9

Cars are passing a particular point on a bridge. For the purposes of this problem we will assume that each car is 3,5 metres long and that there is a distance of d metres between each car. The cars are all travelling at v km/h.

(a) Show that the number of cars passing through the point on the bridge each hour (i.e. the *flow rate*) is given by:

$$F = \frac{1000v}{d + 3,5}$$

(4)

Each car can be thought of as being 3,5 + d metres long. In 1 hour, v km of cars pass through the bridge i.e. 1000v metres. Therefore the number of cars is: $\frac{1000v}{d + 3,5}$

(b) To ensure safe driving, we insist that cars keep a safe following distance, represented by the value d. Let us assume that a car travelling at v km/h requires a following distance of $d = 0,006v^2$.

Now find the velocity that maximises the flow rate of traffic over the bridge. (10)

$$F = \frac{1000v}{0,006v^2 + 3,5}$$

$$\frac{dF}{dv} = \frac{\left(0,006v^2 + 3,5\right).1000 - 1000v\left(0,012v\right)}{\left(0,006v^2 + 3,5\right)^2}$$

$$\therefore 6v^2 + 3500 - 12v^2 = 0$$

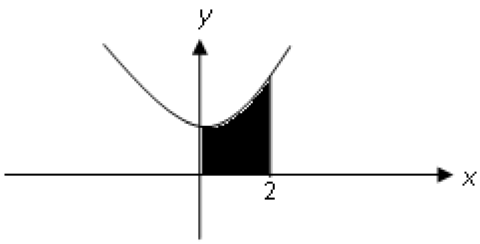
$$v^2 = \frac{3500}{6}$$

$$v = 24,15\,km/h$$

[14]

Question 10

Use Riemann Sums to determine the shaded area, bounded by the curve $y = 2x^2 + 1$ the axes and the line $x = 2$.



[10]

$$f(x) = 2x^2 + 1$$

$$\Delta = \frac{2}{n}$$

$$f\left(\frac{2i}{n}\right) = 2\left(\frac{2i}{n}\right)^2 + 1 = \frac{8i^2}{n^2} + 1$$

$$Area = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}.\left(\frac{8i^2}{n^2} + 1\right)$$

$$= \lim_{n\to\infty}\left[\frac{16}{n^3}\sum i^2 + \frac{2}{n}\sum 1\right]$$

$$= \lim_{n\to\infty}\left[\frac{16}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + \frac{2}{n}\times n\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{16}{3} + \frac{16}{n} + \frac{8}{3n^2}\right) + 2\right]$$

$$= \frac{22}{3}\ units^2$$

Question 11

Integrate the following functions:

(a)

$$\int x.\cos\left(5x^2\right)dx$$

$$=\frac{1}{10}\int 10x.\cos\left(5x^2\right)\,dx$$

$$=\frac{1}{10}\sin\left(5x^2\right)+C$$

(4)

(b)

$$\int \frac{1}{x^2\left(1+\frac{1}{x}\right)^3}\,dx$$

$$=-\int -x^{-2}\left(1+x^{-1}\right)^{-3}\,dx$$

$$=-\frac{\left(1+x^{-1}\right)^{-2}}{-2}+C$$

$$=\frac{1}{2\left(1+\frac{1}{x}\right)^2}+C$$

(8)

(c) Given the general formula: $\int \frac{1}{a^2-x^2}\,dx=\frac{1}{2a}\ln\frac{a+x}{a-x}+C$

find, without the use of a calculator, the value of: $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{9-4x^2}\,dx$ (8)

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{9-4x^2}\,dx$$

$$=\frac{1}{4}\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\frac{9}{4}-x^2}\,dx$$

$$=\frac{1}{4}\times\frac{1}{2\left(\frac{3}{2}\right)}\left(\ln\frac{\frac{3}{2}+\frac{1}{2}}{\frac{3}{2}-\frac{1}{2}}-\ln\frac{\frac{3}{2}+\frac{1}{4}}{\frac{3}{2}-\frac{1}{4}}\right)$$

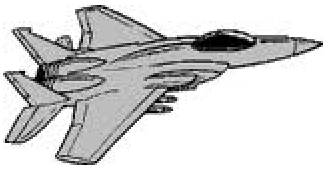
$$=\frac{1}{12}\left(\ln 2-\ln\tfrac{7}{5}\right)$$

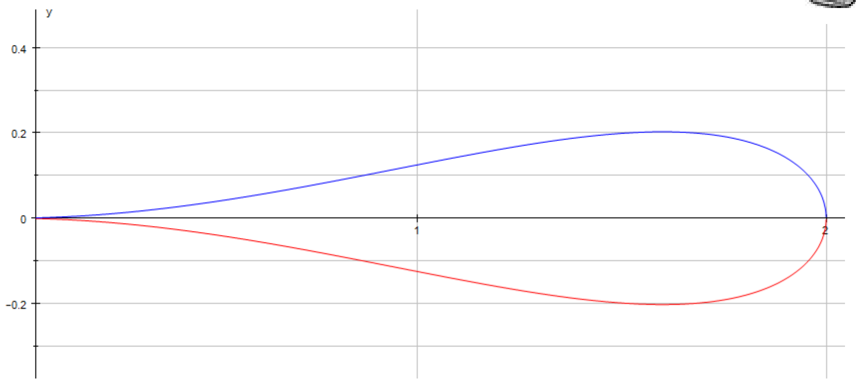
$$=\frac{1}{12}\ln\tfrac{10}{7}$$

[20]

Question 12

The fuel tank on the wing of a jet is formed by rotating the region bounded by the curve $y = \frac{1}{8}x^2.\sqrt{2-x}$ about the x-axis between $x = 0$ and $x = 2$, where the units are measured in metres.





Determine the volume of the fuel tank. **[12]**

$$V = \pi \int_0^2 \tfrac{1}{64} x^4.(2-x)\, dx$$

$$= \frac{\pi}{64}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2$$

$$= \frac{\pi}{64}\left(\frac{64}{5} - \frac{64}{6}\right)$$

$$= \frac{\pi}{30}\ units^3$$

Section B – Finance (100 marks)

Question 1

An investment of P now is worth four times as much in 8 years' time. Calculate:

(a) The effective annual interest rate. (8)

$$4P = P(1+i)^8$$
$$\therefore 4 = (1+i)^8$$
$$\therefore i = \sqrt[8]{4} - 1$$
$$= 0{,}1892$$
$$= 18{,}92\%$$

(b) The nominal annual rate if compounding occurs quarterly. (5)

$$0{,}1892 = \left(1+\frac{r}{4}\right)^4 - 1$$

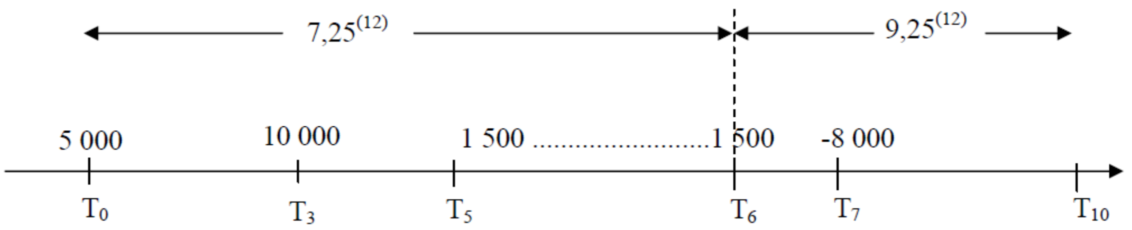
$$\therefore r = 17{,}71\%$$

[13]

Question 2

- Sophie deposits a lump sum of R5 000 in an account giving 7,25% interest per annum compounded monthly.
- After 3 years she adds a further R10 000 to the account.
- During the sixth year she makes 12 **monthly** deposits of R1 500, starting at the end of the first month.
- At the end of the 6th year the interest rate increases to 9,25% per annum.
- She withdraws R8 000 after 7 years

(a) Draw a time-line to represent the information given over a ten-year time period. (8)



(b) Determine the amount that can be withdrawn after a total of ten years. (16)

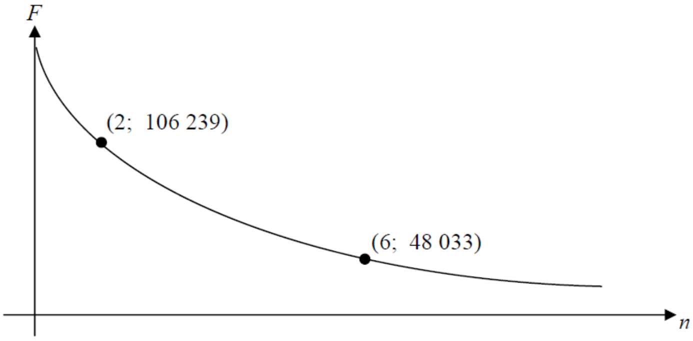
$$F = 5000\left(1+\frac{0,0725}{12}\right)^{72}(1+0,0925)^4 + 10000\left(1+\frac{0,0725}{12}\right)^{36}(1+0,0925)^4 + 1500\left[\frac{(1+\frac{0,0725}{12})^{12}-1}{\frac{0,0725}{12}}\right](1+0,0925)^4 - 8000(1+0,0925)^3$$

F = R44765,77

[24]

Question 3

The depreciation of a vehicle is represented by the graph below where F is the current value of the car (to the nearest rand) and n is the age of the car in years.



(a) What is the significance of the y-intercept of the graph? (2)

This is the original value of the car.

(b) According to this model, will the value of the vehicle ever become 0. Explain. (3)

This is an exponential curve which has an asymptote F = 0, therefore F will never reach 0.

(c) Calculate: (i) the annual rate of depreciation (11)

(ii) the original value of the car. (4)

$106239 = P(1-i)^2$ ....(1)

$48033 = P(1-i)^6$ ....(2)

Divide (2) by (1):

$(1-i)^4 = \frac{48033}{106239}$

$\therefore i = 0{,}18$

$\therefore 18\%\ \ p.a.$

$\therefore P = R158\,000$

[20]

Question 4

A loan is taken to assist in the purchase of a town house property. The interest on the outstanding balance is 13% per annum compounded monthly.

(a) If the interest payable for the first month is R780, show that the amount of the original loan was R72 000. (4)

$$\frac{0{,}13}{12} \times P = 780$$
$$\therefore P = R72\,000$$

(b) If the loan is to be repaid over a period of 20 years, calculate the monthly payment. (9)

$$72000 = x\left[\frac{1-\left(1+\frac{0{,}13}{12}\right)^{-240}}{\frac{0{,}13}{12}}\right]$$

x = R843,53

(c) It is possible to repay the loan over a longer period. How would you advise a client wishing to extend the repayment time of the loan? What are the advantages and pitfalls? (6)

[19]

Advantage - lower monthly payments.
Disadvantages - pay more in the end i.e. more interest.
Rather be debt-free quicker.

Question 5

(a) An iterative formula for calculating the value of a sinking fund is:

$F_{n+1} = 1{,}01 \times F_n + 250$ where: $F_0 = 0$

(i) Write down F_1, F_2 and F_3. (6)

$F_1 = 250$
$F_2 = 502.5$
$F_3 = 757.525$

(ii) Find the effective annual interest rate. (8)

Monthly rate = 0,01 = 1%
Nominal annual rate = 12%

$$\text{Effective annual rate} = \left(1 + \frac{0{,}12}{12}\right)^{12} - 1$$
$$= 12{,}68\%$$

(b) Solve for a and b if the sequence -3; 10; 8; -6; -11; .... is generated by the recursive formula: (10)

$$T_n = \frac{T_{n-1}}{a} + b.T_{n-2}$$

[24]

$$8 = \frac{10}{a} - 3b$$
$$\therefore 8a = 10 - 3ab$$

$$-6 = \frac{8}{a} + 10b$$
$$\therefore -6a = 8 + 10ab$$

$$\frac{10 - 8a}{3} = \frac{-6a - 8}{10}$$
$$\therefore 100 - 80a = -18a - 24$$
$$\therefore 62a = 124$$
$$\therefore a = 2$$
$$\therefore b = -1$$

END OF EXAMINATION

INFORMATION SHEET

Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

$$z = a + bi \qquad z^* = a - bi$$

$$\ell n\, A + \ell n\, B = \ell n\,(AB) \qquad \ell n\, A - \ell n\, B = \ell n \left(\frac{A}{B}\right)$$

$$\ell n\, A^n = n\, \ell n\, A \qquad \log_a x = \frac{\log_b x}{\log_b a}$$

Calculus

$$Area = \lim_{n \to \infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} f\left(x_i\right)$$

$$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$$

$$f\,'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\int f\,'(g(x)).g\,'(x)\,dx = f(g(x)) + c$$

$$\int f(x).g\,'(x)dx = f(x).g(x) - \int g(x).f\,'(x)\ dx + c$$

$$x_{r+1} = x_r - \frac{f(x_r)}{f\,'(x_r)}$$

$$V = \pi \int_a^b y^2 dx$$

Function	Derivative
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x.\tan x$
$\text{cosec}\, x$	$-\text{cosec}\, x.\cot x$
$f(g(x))$	$f'(g(x)).g'(x)$
$f(x).g(x)$	$g(x).f'(x)+f(x).g'(x)$
$\dfrac{f(x)}{g(x)}$	$\dfrac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2}$

$A=\frac{1}{2}r^2\theta$ $\qquad$ $s=r\theta$

In $\triangle ABC$: $\qquad$ $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$$a^2=b^2+c^2-2bc.\cos A$$

$$Area=\frac{1}{2}ab.\sin C$$

$\sin^2 A+\cos^2 A=1$ $\qquad$ $1+\tan^2 A=\sec^2 A$ $\qquad$ $1+\cot^2 A=\text{cosec}^2 A$

$\sin(A\pm B)=\sin A.\cos B\pm\cos A\sin B$ $\qquad$ $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\sin 2A=2\sin A\cos A$ $\qquad$ $\cos 2A=\cos^2 A-\sin^2 A$

$$\sin A.\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$\sin A.\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\cos A.\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

Finance & Modelling

$F=P(1+in)$ $\qquad$ $F=P(1-in)$ $\qquad$ $F=P(1+i)^n$ $\qquad$ $F=P(1-i)^n$

$F=x\left[\dfrac{(1+i)^n-1}{i}\right]$ $\qquad$ $P=x\left[\dfrac{1-(1+i)^{-n}}{i}\right]$ $\qquad$ $r_{eff}=\left(1+\dfrac{r}{k}\right)^k-1$