St John's College

LUX VITA CARITAS

UPPER V

Advanced Programme Mathematics

July 2012 **Time: 3 hours** **Marks: 300**

Examiner: K Jacobs
Moderator: W Young

INSTRUCTIONS AND INFORMATION

Read the following instructions carefully

1. This paper consists of 12 pages, including an information sheet. Please make sure your paper is complete.

2. Read the questions carefully.

3. Plan your time carefully.

   Section A: Calculus & Algebra (210 marks)
   Section B: Financial Mathematics (90 marks)

4. <u>**Section A and Section B must be answered in separate booklets**</u>**.**

5. Answer all the questions.

6. Number your answers exactly as the questions are numbered.

7. You may use an approved non-programmable and non-graphical calculator, unless otherwise stated.

8. Round off your answers to one decimal digit, where appropriate.

9. All the necessary and reasonable working details must be clearly shown.

10. It is in your own interest to write legibly and to present your work neatly.

Section A – Calculus and Algebra (210 marks)

Question 1

Prove by induction that $\sum_{i=1}^{n} \frac{1}{(i+1)(i+2)} = \frac{n}{2(n+2)}$, $n \in$

[12]

Question 2

Determine $g(f(x))$ if $f(g(x)) = \frac{1}{x+3} + x^2 + 6x + 9$ and $g(g(x)) = x + 6$

[6]

Question 3

(a) Solve for x: $3(\ln x)^2 + \ln x - 1 + \frac{1}{3(\ln x)^2 + \ln x - 3} = 0$ (9)

(b) For which value(s) of x will the following function be defined:

$$\ln\left(\frac{6x+1}{(2x-3)^2}\right)$$ (4)

(c) Given $p(x) = x^3 + ax^2 + bx - 6$ with a zero at $x = 1 + i$.
Determine the values of a and b. (8)

[21]

Question 4

(a) Sketch the graph of $h(x) = \left|x^2 - 6x - 7\right| - 9$, showing all intercepts and the turning point clearly. (11)

(b) Algebraically, determine the value(s) of x where $h(x)$ intercepts the graph $m(x) = -x + 20$ (6)

(c) How many roots does $h(x)$ have? (2)

(d) Using your graph, determine the value of k if $j(x) = h(x) + k$ has exactly 3 solutions. (3)

[22]

Question 5

If an object is put in an environment at a fixed temperature, ***A*** (the "ambient temperature"), then it's temperature, ***T*** (in degrees Fahrenheit), at a time ***t*** (in minutes) is modelled by Newton's Law of cooling: $T = A + Ce^{-kt}$ where ***k*** is positive constant.

Assume that a hot cup of tea (at 160^0F) is left to cool in a 75^0F room, and that it takes 14 minutes for it to reach 100^0F.

(a) Using the above equation, explain why over a long period of time, the temperature approximates the ambient temperature *A*. (2)

(b) Write down the value of *A*. (2)

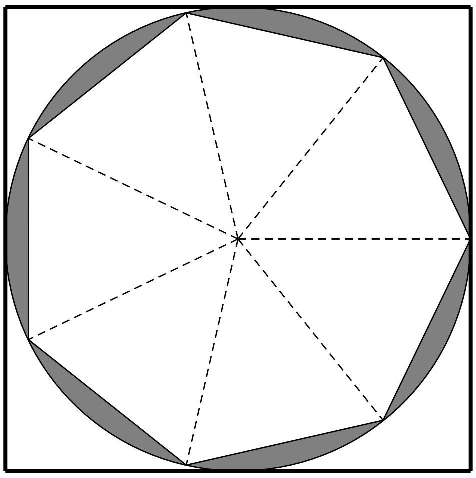
(c) Determine the values of *C* and *k*. (6)

(d) What is the temperature of the tea after 20 minutes?
Convert your answer to degrees celcius, using the following conversion:
$^0C = \frac{5}{9}(^0F - 32)$ (4)

[14]

Question 6

A pattern on a square tile is created from 7 equal sectors as shown below.
The circle is inscribed in the square.
The **total shaded area** in the tile below measures 8,6 cm^2.



Determine the area of the square tile.

[8]

Question 7

(a) (1) Prove $$\frac{1}{\sec A-1}+\frac{1}{\sec A+1}=2\cos A.\operatorname{cosec}^2 A$$ (6)

(2) Hence determine $$\lim_{A\to 0}\left(\frac{1}{\sec A-1}+\frac{1}{\sec A+1}\right)A^2$$ (5)

(b) Determine $$\lim_{x\to 1}\frac{x^3-1}{\sqrt{x}-1}$$ (6)

[17]

Question 8

$$g(x)=\begin{cases} 2x+1 & \text{if } x\le p \\ -\frac{1}{4}(x^3-2x^2-7x-4) & \text{if } x>p \end{cases}$$

(a) For which value(s) of p is $g(x)$ continuous at p? (7)

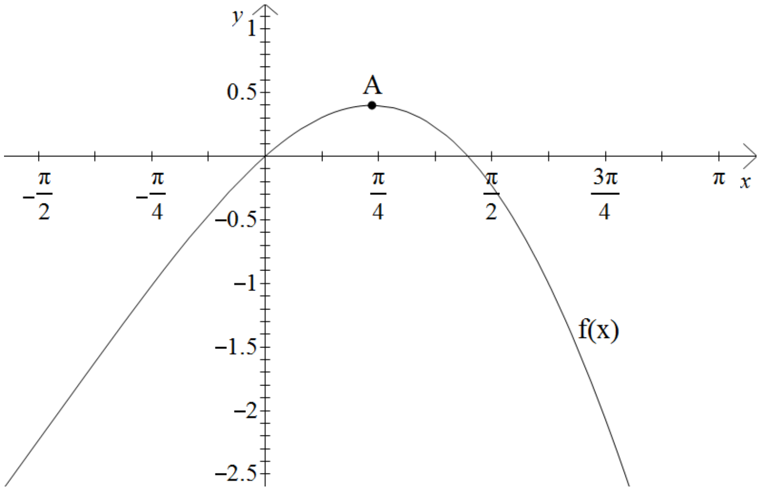
(b) For which value(s) of p is $g(x)$ differentiable at p? (8)

[15]

Question 9

The function $f(x)$ is defined by $f(x)=\sin x-\frac{1}{2}x^2$ and is sketched below.

Using Newton's method, determine the coordinate of A, the local maximum of $f(x)$, to four decimal places.



[12]

Question 10

Create a function whose graph has the indicated characteristics:
(The answers are not unique).

(a) A vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 0$. (4)

(b) One asymptote which is vertical at $x = -3$ and a removable discontinuity at $x = 3$. (4)

(c) A vertical asymptote at $x = 0$ and an oblique asymptote at $y = -x$. (6)

[14]

Question 11

(a) $y = \sqrt{4x^2 + 1}$

(1) Show that $\frac{dy}{dx} = \frac{4x}{y}$ (4)

(2) Hence or otherwise show that $\frac{d^2y}{dx^2} = \frac{4}{y} - \frac{16x^2}{y^3}$ (6)

(b) (1) Show that $\frac{dy}{dx} = \frac{-(x-a)}{y-b}$ for the circle $(x-a)^2 + (y-b)^2 = r^2$ (6)

(2) Hence determine the gradient of the tangent to the circle $(x+5)^2 + (y-3)^2 = 100$ passing through the point (1; -5) (4)

[20]

Question 12

Integrate the following functions:

(a) $\int (ax+b)^2 \, dx$ (6)

(b) $\int 2x \sec^2 x^2 dx$ (4)

(c) $\int x \sin 3x \, dx$ (8)

(d) $\int \frac{\cos 3x}{\cos ec 5x} \, dx$ (6)

[24]

Question 13

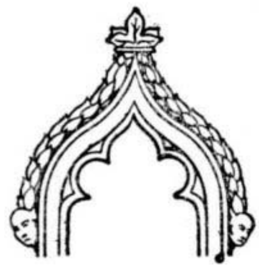
Simpson’s Rule is as follows:

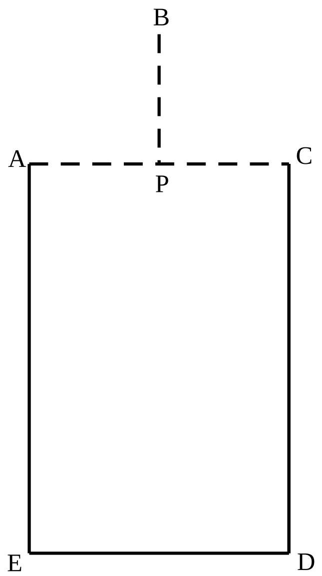
$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

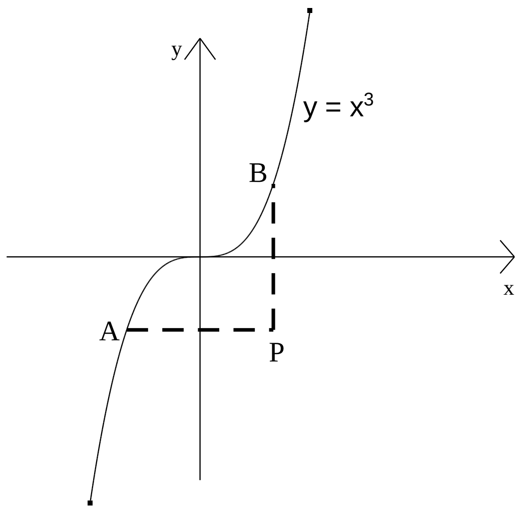
Using this rule, determine $\int_1^7 \frac{1}{2x+1}\,dx$ **[8]**

Question 14

A Gothic (ogee) window (similar to the one shown alongside) is to be constructed as indicated.
AB is part of the curve $y = x^3$;
BP $\perp$ AC; AP = PC = BP; DE = 4 units; CD = 6 units







Determine:

(a) the area of ABP (6)

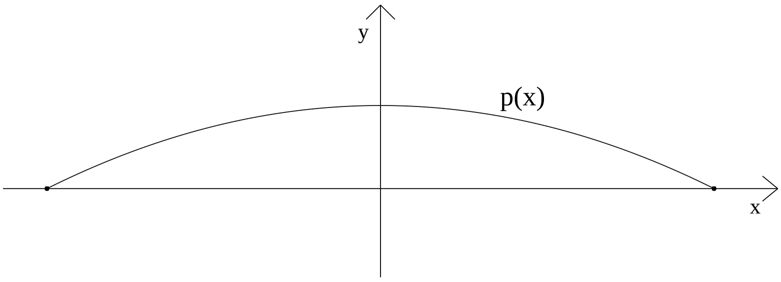
(b) the area of the window ABCDE (3)

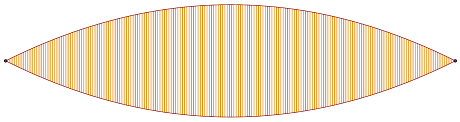
[9]

Question 15

A floatation device called a pontoon is to be made in the shape shown below.

The pontoon is created by rotating the graph of $p(x) = 1 - \frac{x^2}{16}$ about the x-axis, where *x* and *y* are measured in metres.





Determine the volume of the pontoon.

[8]

Don't forget to start a new booklet!

Section B – Financial Mathematics (90 marks)

Question 1

Murray opens a savings account and deposits R5 000 into the account immediately. Five months later he invests a further R*x* into the account. The interest rate for the first four months is 18% per annum compounded monthly, 21% per annum compounded monthly for the next two months and 24% per annum compounded monthly thereafter. The investment has a future value of R100 000 at the end of the 8th month. Calculate the value of *x*.

[15]

Question 2

An amount *P* was invested at *i* % per annum compounded monthly for two years. The accumulated amount, R1 196,41 was reinvested at *i* % per annum compounded quarterly. After a further 3 years, the investment was worth R1 609,04. Determine *i* and evaluate *P*.

[10]

Question 3

A contractor buys a truck for R800 000. The value of the truck depreciates by 25% per annum on a reducing balance. This truck will need to be replaced at the end of 4 years. The value of a new truck is expected to appreciate by 16% per annum.

(a) Calculate the resale value of the truck in 4 years' time. (4)

(b) What should be the value of a sinking fund that needs to be set up to pay for the new truck, if the old truck is used as a trade-in? (6)

(c) Monthly payments are made into a sinking fund account which earns interest at 12% per annum compounded monthly. Payments commence in 3 months' time, and finish in 4 years' time. Calculate the monthly payment. (5)

[15]

Question 4

A loan of R8 000 is repaid by means of equal payments of R650 every quarter, and a final payment, one quarter after the last payment of R650. Interest is 15,6% per annum compounded quarterly.

(a) Find the number of payments <u>and</u> the value of the final payment if the first payment is made three months from the granting of the loan. (14)

(b) If the loan is to be repaid with the first payment being made after three years from the granting of the loan, determine the number of payments that now need to be made. (10)

[24]

Question 5

Pete will matriculate in 2012 and wants to study a BCom for three years.

The fees are as follows:

- The cost of the first year of a BCom in 2013, including books, is R34 000. This is payable at the beginning of every year.
- The fees for each of the successive two years will increase by 9% p.a.

Pete will need to get a student loan to pay for his studies.
A student loan from a bank involves 2 parties:

- The person signing on your behalf stands as surety and is expected to repay the interest while you are studying.
- The student repays the loan (without interest) as soon as he starts to work.
  Repayment periods are equal to 1,5 years for every year's assistance granted.

Assume a fixed interest rate of 10,5% p.a. compounded monthly.

(a) Calculate the total interest on the loan at the end of the three years of study that the person standing as surety will have to pay. (11)

(b) Pete's parents stand as surety. If their first payment occurs at the end of January 2013, and their last payment at the end of December 2015, calculate their monthly payment to the bank. (5)

(c) Pete starts working in January 2016. Calculate his monthly payment to the bank if his first payment is at the end of January 2016. (6)

(d) How much did Pete and his parents end up paying for his degree? (4)

[26]

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INFORMATION SHEET

Algebra

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

$z = a + bi$ $\qquad$ $z^* = a - bi$

$\ell n\, A + \ell n\, B \ = \ \ell n\,(AB)$ $\qquad$ $\ell n\, A - \ell n\, B \ = \ \ell n \left(\frac{A}{B}\right)$

$\ell n\, A^n \ = \ n\ \ell n\, A$ $\qquad$ $\log_a x \ = \ \frac{\log_b x}{\log_b a}$

Calculus

$$Area = \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} f(x_i)$$

$$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\int f'(g(x)).g'(x)\, dx = f(g(x)) + c$$

$$\int f(x).g'(x) dx = f(x).g(x) - \int g(x).f'(x)\ dx + c$$

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

$$V = \pi \int_a^b y^2 dx$$

Function	Derivative
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x.\tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x.\cot x$
$f(g(x))$	$f'(g(x)).g'(x)$
$f(x).g(x)$	$g(x).f'(x)+f(x).g'(x)$
$\dfrac{f(x)}{g(x)}$	$\dfrac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2}$

$A=\frac{1}{2}r^2\theta$ $\qquad s=r\theta$

In △ABC: $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$$a^2=b^2+c^2-2bc.\cos A$$

$$Area=\frac{1}{2}ab.\sin C$$

$\sin^2 A+\cos^2 A=1$ $\qquad 1+\tan^2 A=\sec^2 A$ $\qquad 1+\cot^2 A=\operatorname{cosec}^2 A$

$\sin(A\pm B)=\sin A.\cos B\pm\cos A\sin B$ $\qquad \cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$\sin 2A=2\sin A\cos A$ $\qquad \cos 2A=\cos^2 A-\sin^2 A$

$$\sin A.\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$\sin A.\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\cos A.\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

Finance & Modelling

$F=P(1+in)$ $\qquad F=P(1-in)$ $\qquad F=P(1+i)^n$ $\qquad F=P(1-i)^n$

$F=x\left[\dfrac{(1+i)^n-1}{i}\right]$ $\qquad P=x\left[\dfrac{1-(1+i)^{-n}}{i}\right]$ $\qquad r_{eff}=\left(1+\dfrac{r}{k}\right)^k-1$