GRADE 12 EXAMINATION
NOVEMBER 2009

ADVANCED PROGRAMME MATHEMATICS

Time: 3 hours — 300 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 19 pages, an answer sheet for question 4.1 and an Information Booklet of 4 pages (i – iv). Please check that your question paper is complete. Please remove the insert and answer sheet from the middle of the question paper.

2. This question paper consists of FOUR Modules:

   MODULE 1: CALCULUS AND ALGEBRA (200 marks) is compulsory.

   Choose ONE of the THREE Optional Modules:

   MODULE 2: STATISTICS (100 marks) OR
   MODULE 3: FINANCE AND MODELLING (100 marks) OR
   MODULE 4: MATRICES AND GRAPH THEORY (100 marks)

3. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.

4. All necessary calculations must be clearly shown and writing should be legible.

5. Diagrams have not been drawn to scale.

6. Write all your answers in the separate Answer Book provided.

7. Round off your answers to two decimal digits, unless otherwise indicated.

MODULE 1 CALCULUS AND ALGEBRA

QUESTION 1

Use Mathematical induction and prove that $3^n + 3^{n+1} + 3^{n+2}$ is divisible by 13, for all $n \in$ N. (13)

13 marks

QUESTION 2

2.1 Given $f(x) = e^{x+2} - 1$

(a) Determine the equation of $f^{-1}(x)$, the inverse of $f(x)$. (4)

(b) Hence sketch the graph of $f^{-1}(x)$, clearly indicating the values of the intercepts with the axes correct to 1 decimal place, and asymptotes. (9)

2.2

(a) If $\sqrt{x + iy} = a + ib$, show that $x = a^2 - b^2$ and $y = 2ab$. (5)

(b) Hence find x and y if $\sqrt{x + iy} = 5i - 12$. (4)

2.3 If $f(x) = \dfrac{1}{4 - x^2}$, decompose $f(x)$ into partial fractions. (9)

31 marks

QUESTION 3

3.1 It is given that $x = 1 - 2i$ and $x = 1 + 2i$ are both zeros of $g(x)$. Use this information to show that $x^2 - 2x + 5$ is a factor of $g(x)$. (4)

3.2 Given: $g(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$

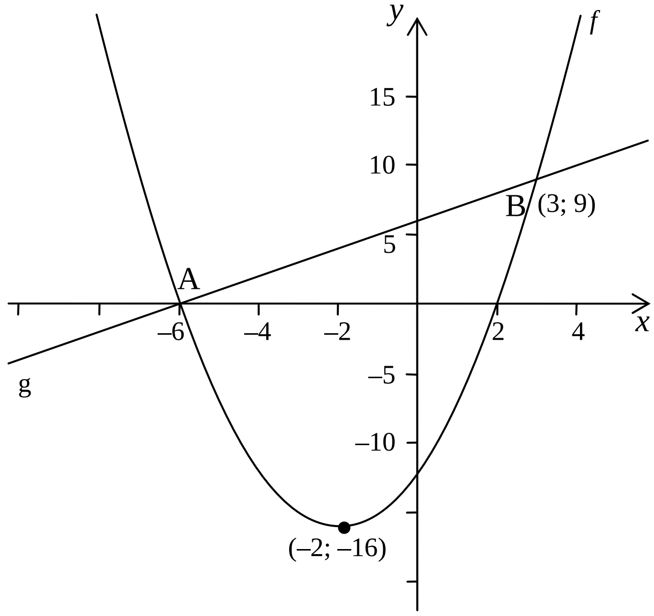
According to Gauss, a polynomial of degree n has n zeros. Therefore $g(x)$ has four zeros. Determine the other two zeros of $g(x)$ if $x = 1 - 2i$ and $x = 1 + 2i$ are both zeros of $g(x)$.

(11)

15 marks

QUESTION 4

The sketch shows the graphs of $g(x) = x + 6$ and $f(x) = x^2 + 4x - 12$, which intersects at $A(-6; 0)$ and $B(3; 9)$.



4.1 Use the information given on the sketch to draw, ***on the answer sheet provided***, graphs of $h(x) = |x + 6|$ and $j(x) = |x^2 + 4x - 12|$. (6)

4.2 These two graphs (*h* and *j*) intersect at the same points as $g(x)$ and $f(x)$ as well as at a third point. Calculate, without the use of a calculator, the coordinates of the other point of intersection of $h(x)$ and $j(x)$. (7)

13 marks

QUESTION 5

Given the function: $h(x) = \begin{cases} 2x^3 + x^2 - 3, & x < -1 \\ -x + k, & x \geq -1 \end{cases}$

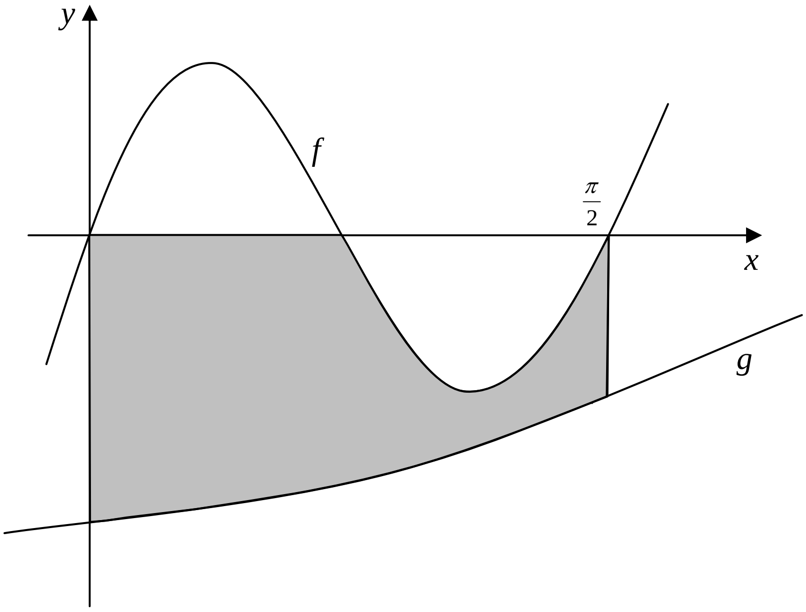
5.1 Find the real value of k if $h(x)$ is continuous at $x = -1$. (6)

5.2 Is $h(x)$ differentiable at $x = -1$? Assume $k = -5$ and justify your answer mathematically. (8)

14 marks

QUESTION 6

In the given sketch, the shaded region is defined on the interval $\left[0; \frac{\pi}{2}\right]$ and is bounded by the functions $f(x) = 3\sin 4x$ and $g(x) = \frac{1}{2}(x+1)^2 - 6$. The two functions do not intersect anywhere on this interval.



6.1 Show that the maximum distance between the two graphs in this interval can be obtained by solving the equation $12\cos 4x = x + 1$. (6)

6.2 Use Newton's method to write down a recursive equation that can be used to solve the equation in 6.1. Hence, taking $x = 0{,}5$ as an initial value, determine the answer correct to five decimal places. (9)

15 marks

QUESTION 7

A certain function, $f(x)$, has the following characteristics:

- Exactly two roots: $x = 1$ and $x = -2$
- Exactly two vertical asymptotes with equations $x = 7$ and $x = -1$
- A horizontal asymptote at $y = 2$

Determine a possible expression for $f(x)$. (It is not necessary to simplify your answer.) (6)

6 marks

QUESTION 8

The functions f, g and h are defined as follows:

$$f(x) = \sin(\tan(2x))$$

$$g(x) = x^{\frac{2}{3}}\left(x + \sqrt[4]{x}\right)$$

$$h(x) = \frac{2x}{\cos 2x}$$

8.1 Determine $f'(x)$, $g'(x)$ and $h'(x)$. (10)

8.2 Arrange $f'(\pi)$, $g'(1)$ and $h'\left(\frac{\pi}{2}\right)$ in descending order. (4)

14 marks

QUESTION 9

9.1 Find an expression for $\frac{dy}{dx}$ in terms of x and y for the implicitly defined curve $x^2 - 4xy + 4y + 8 = 0$. (9)

9.2 Hence determine the co-ordinates of the stationary points on this curve. (9)

18 marks

QUESTION 10

Damon is finding $\int x\cos 2x\,dx$. He uses integration by parts, but he comes to a stop after a few lines of working **as he realises that he can't solve the problem**. The last line of his working is shown:

$$\int x\cos 2x\,dx = \frac{1}{2}x^2\cos 2x + \int x^2.\sin 2x dx$$

10.1 What did Damon do to arrive at this step? (4)

10.2 This is, however, not the best method to use in solving this problem. What suggestion would you make to Damon to solve this problem in a simpler way? (2)

10.3 Using your suggestion, what answer will Damon get? (6)

12 marks

QUESTION 11

11.1 Determine the following integrals: (You do not have to simplify your answer.)

(a) $\int \frac{1}{\sqrt{1-5x}}\,dx$ (5)

(b) $\int 2\sin 4x\cos 5x\,dx$ (6)

11.2

(a) Prove the following identity: $\frac{1}{1+\sin x} + \frac{1}{1-\sin x} = 2\sec^2 x$ (6)

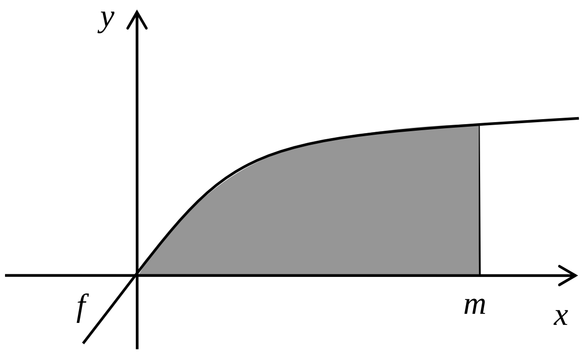
(b) Hence determine the **smallest positive** value of a if

$$\int_0^a \left(\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right)dx = \frac{2\sqrt{3}}{3}$$ (7)

24 marks

QUESTION 12

The sketch shows the graph of $f(x)=\dfrac{x}{\sqrt{2x^2+1}}$ which cuts the axes at the origin. The shaded region is the area between the graph, the x-axis and the line $x = m$.



12.1 Determine the area of the shaded region in terms of m. (9)

12.2 Calculate this area if $m = 2$. (2)

12.3 Set up, but do not integrate or calculate, an expression that represents the volume of a solid that will be formed if the shaded region is rotated around the x-axis. (4)

15 marks

QUESTION 13

$f(a)$	1
$f'(a)$	2
$g(a)$	?
$g'(a)$	6
$h(a)$	$-\frac{1}{3}$
$h'(a)$	?

Assume that the functions $f(x)$, $g(x)$ and $h(x)$ are continuous and differentiable for all real values, as are their derivatives. In the table alongside, the functions are evaluated at a point where $x = a$. However, two of the values are omitted.

It is also given that $h(x)=\dfrac{f(x)}{g(x)}$

Showing all the calculations that lead to your answers, calculate the value of:

13.1 $g(a)$ (3)

13.2 $h'(a)$ (7)

10 marks

Total for Module 1: 200 marks

MODULE 2 STATISTICS

QUESTION 1

Gas that is produced from biological fermentation is sold with the claim that the methane content is normally distributed with a mean of 70% and a standard deviation of 1%. A random sample of 8 gas canisters gave the following methane contents expressed as a percentage:

Sample	1	2	3	4	5	6	7	8
% methane	64	65	75	67	65	74	75	69

1.1 Conduct a hypothesis test at the 5% significance level to determine whether, based on this sample, it is fair to claim that the mean methane content is 70%. (10)

1.2 Based on this sample determine a 99% confidence interval for the mean methane content. (8)

18 marks

QUESTION 2

To test the effect of alcohol on reaction time, a sample of 14 students was selected. The students performed a test measuring their reaction time (before they had consumed any alcohol) and were given an initial score, I , out of 100. They were then given 5 units of alcohol and their reaction time was retested 30 minutes later, when they were given a final score, F, out of 100.

A decrease in score indicates a decrease in reaction time.

Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14
I	90	87	93	86	81	84	70	87	86	91	89	88	90	92
F	75	72	76	70	68	73	56	80	73	79	79	69	77	80

Give your answers to the following correct to 4 decimal places, where appropriate.

2.1 (a) Determine the correlation coefficient, r, for this data. (6)
(b) What does this tell us about the nature and strength of the linear relationship between I and F? (3)

2.2 (a) Determine the equation of the least squares regression line of F on I. (5)
(b) Two further students, Prashail and Shoso, are tested and their initial scores are 85 and 64. Predict what their scores would be after they had 5 units of alcohol. (5)
(c) Which prediction made in 2.2 (b) would be more reliable? Give an explanation to support your claim. (3)

2.3 (a) Determine the mean and standard deviation of the initial scores (excluding Prashail and Shoso). (3)

(b) Which student, from the table above, could be considered an outlier? (2)

(c) If this outlier were removed from the data, how would the strength of the straight-line relationship between the scores change? Give an explanation to support your answer. (5)

32 marks

QUESTION 3

3.1 A manufacturer is producing table tennis balls and 4% of all table tennis balls are damaged during manufacturing. The table tennis championship organisers receive a set of 500 balls for a tournament.

(a) How many balls would you expect to be defective in a set of 500 balls? (2)

(b) It would take too long to check all the balls so the organisers decided to randomly inspect 20 balls. If this sample has more than one defective ball the set of 500 balls will be returned. What is the probability that the set of 500 balls will be accepted? (10)

3.2 New table tennis bats are used in each of the twenty matches during the tournament. The organisers have observed that in 15% of matches a bat needs to be replaced during the match. What is the probability that exactly 6 bats will be replaced in the 20 matches of the tournament? (10)

3.3 The length of competitors' arms was recorded and was found to be normally distributed with a mean of 80 cm and a variance of 15 cm. What is the probability that a competitor selected at random has an arm length of more than 90 cm? (8)

30 marks

QUESTION 4

Given that $P(A|B) = \frac{2}{5}$; $P(B|A) = \frac{1}{4}$ and $P(A \cup B) = \frac{11}{20}$

4.1 Find $P(A \cap B)$ in terms of $P(A)$. (5)

4.2 Find $P(A \cap B)$ in terms of $P(B)$. (3)

4.3 Hence, or otherwise, find $P(A)$. (9)

4.4 Find $P(A \cap B)$. (3)

20 marks

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

1.1 Sam has been recording her country's annual inflation figures for the last four years.
2005: 11%
2006: 8,7%
2007: 6,5%
2008: 8,2%
What was the effective annual inflation during this time period? (7)

1.2 Jane decided to buy a car for R125 000. She financed the entire cost through a bank which charged interest at 13% per annum compounded monthly over a 54 month period. Her first payment was exactly one month after purchasing the car.

(a) Calculate her monthly repayments. (7)
(b) What is the balance outstanding on her car immediately after her 24^{th} payment? (8)

22 marks

QUESTION 2

Daniel decides to take out a retirement annuity starting on his 30^{th} birthday, the first payment being made one month later. His premium is fixed at R1 000 per month and the company offering the retirement annuity guarantees an interest rate of 7% interest per annum compounded monthly.

2.1 Daniel's last payment will be on his 60^{th} birthday, at which point he will retire. Assuming that he makes all the payments, what is the total amount that Daniel will have saved for his retirement? (7)

It turns out that, after 5 years of paying into the annuity, Daniel runs into financial difficulties and he misses 8 payments. (Payment numbers 61 to 68). At 10 years (payment number 120) he has sufficient funds available to make a lump sum payment which will make up for the missed payments.

2.2 Determine the value of this lump sum payment (excluding his usual monthly repayment). (10)

On retirement, Daniel withdraws 30% of his annuity in cash and then forms a living annuity, with an interest rate of 8% per annum compounded monthly, with the balance. He assumes that because he has had a very healthy lifestyle he will live to be 85 years old.

2.3 How much will Daniel withdraw in cash? (1)

2.4 Determine Daniel's monthly withdrawals from his living annuity. (8)

26 marks

QUESTION 3

Indera has R20 000 to deposit in a savings account. The bank is offering an interest rate of 9% per annum compounded monthly. Indera will also make a deposit into this account at the end of each month. The first monthly deposit will be R1 000 and this will increase by 1% each month.

3.1 Write a recursive formula to calculate the balance in his savings account. (8)

3.2 How much will Indera have at the end of 3 months? (5)

13 marks

QUESTION 4

The favourite food of the blue whale is a tiny shrimplike crustacean called krill. The blue whale consumes huge quantities of this tiny animal as its main source of food. The maximum sustainable population for krill is 500 tons per acre of ocean. If there are no whales and the ocean is not overcrowded, the krill will increase at 25% per annum.

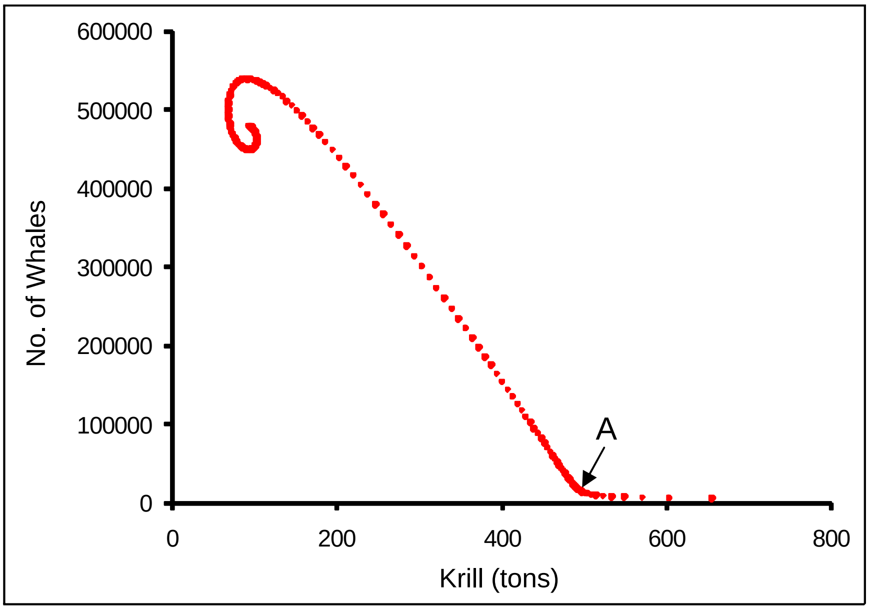
Under ideal conditions blue whales have a life span of 50 years.

If K stands for the krill population, and W for the whale population, then the following equations can be used to model their annual populations:

$$K_{n+1} = K_n + aK_n\left(1 - \frac{K_n}{P}\right) - bK_n.W_n$$
$$W_{n+1} = W_n + f.b.W_n.K_n - cW_n$$

4.1 What are the values of the parameters a and P? (5)
4.2 Explain why $c = 0.02$. (3)

In a certain ocean, the initial population of blue whales has been greatly reduced due to whaling operations and is thus currently only 5 000 whales whilst the krill has reached 750 tons per acre. The values of the parameters b and f are given as $b = 4.3 \times 10^{-7}$ and $f = 500$. The graph showing the change of the krill and blue whale populations in this environment is given below (Assume that all whaling operations cease immediately).

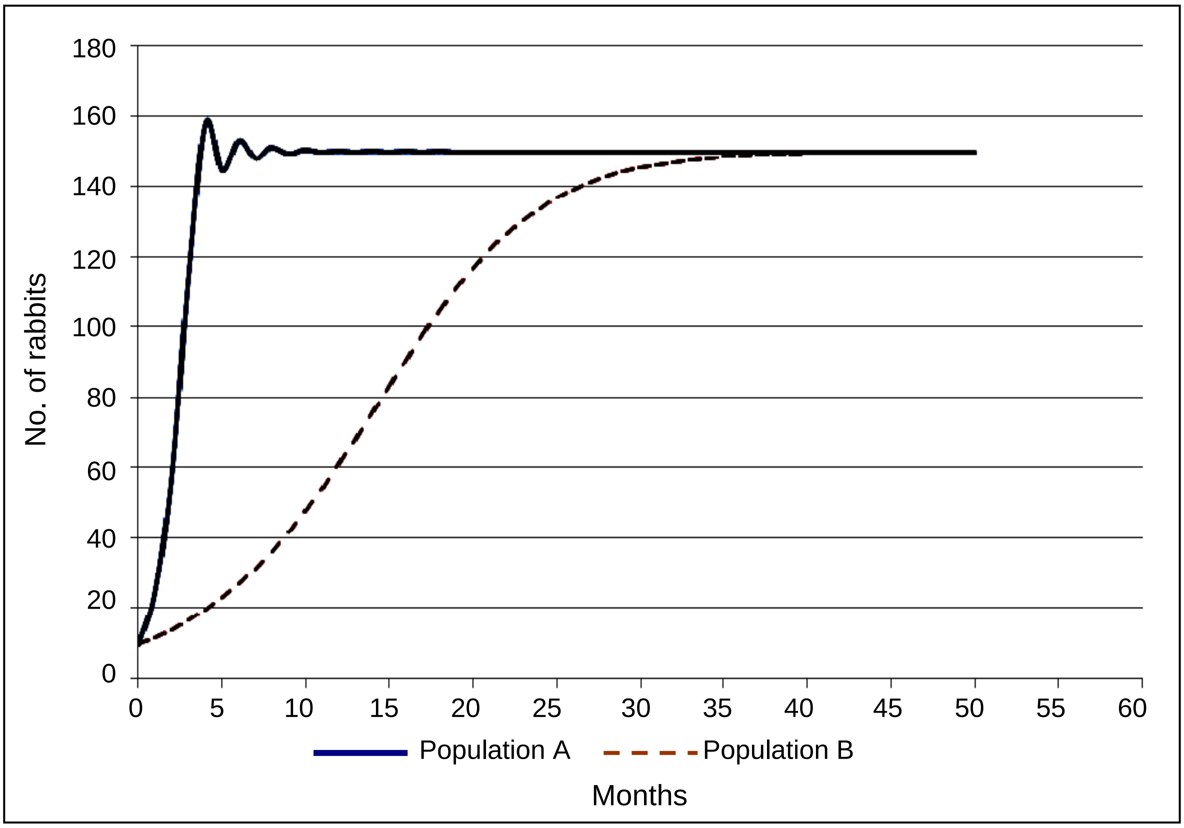


4.3 Explain why the rate of decrease in the krill population changes dramatically at A. (3)
4.4 Estimate from the graph the krill population when the blue whale population is increasing most rapidly. (2)
4.5 Estimate the stable populations for blue whales and krill. (2)
4.6 If the krill's intrinsic growth rate reduces to 10% per annum, determine the new equilibrium point. (12)

27 marks

QUESTION 5

The following graphs show the growth of two independent populations of rabbits. Both have an initial population of 10. One population is fed a special diet to increase their fertility. The carrying capacity of the fields in which they are kept is the same for both groups and there are no predators.



5.1 What is the carrying capacity of the fields? (1)

5.2 Which population received the special diet to increase its fertility? Give a reason for your answer. (2)

5.3 Estimate the intrinsic growth rate of population B. Show calculations to support your answer. (7)

5.4 Explain what is happening to population A between the 4^{th} and the 10^{th} month. (2)

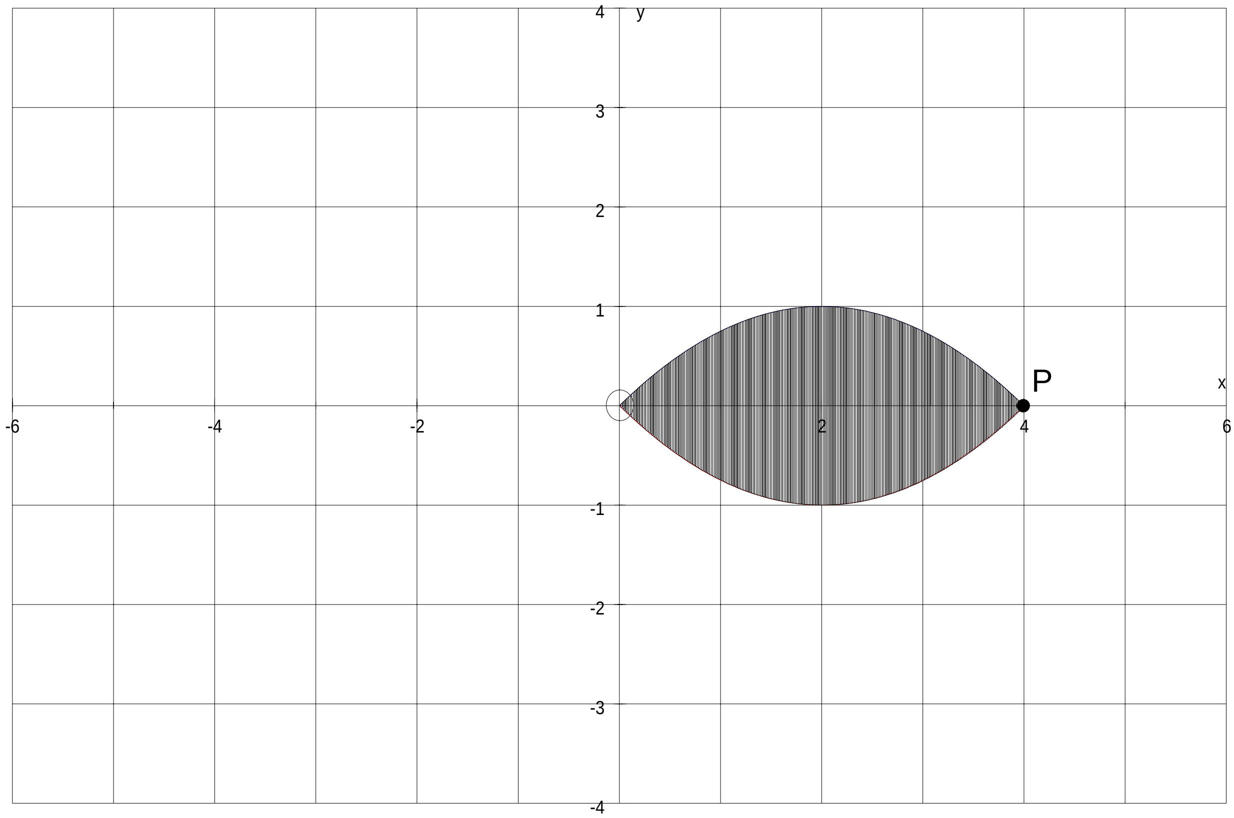
12 marks

Total for Module 3: 100 marks

MODULE 4 GRAPH THEORY AND MATRICES

QUESTION 1

The flowers of several plants have rotational symmetry of degree 5. To make a picture of a flower showing this symmetry a teacher gives the following picture of a petal, then asks the pupils to draw another four petals with the origin as the centre so that the flower has rotational symmetry of degree 5. (It is acceptable for the petals to overlap)



1.1 The first petal that Sam draws is the one next to this petal but rotated in an anti-clockwise direction. Determine the coordinates of the point on this petal that is furthest from the origin. (Answer correct to 2 decimal places) (6)

1.2 Peter thinks the flower will look prettier if he alters the given petal using the following transformation $\begin{pmatrix} 2{,}5 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) How will this change the shape of the original petal? (2)

(b) Find the coordinates of the point P on Peter's petal under this transformation. (3)

1.3 Paul claims that he can make Peter's flower by first finding all the petals using Sam's anti-clockwise rotation, then applying Peter's transformation to all the points in the flower.

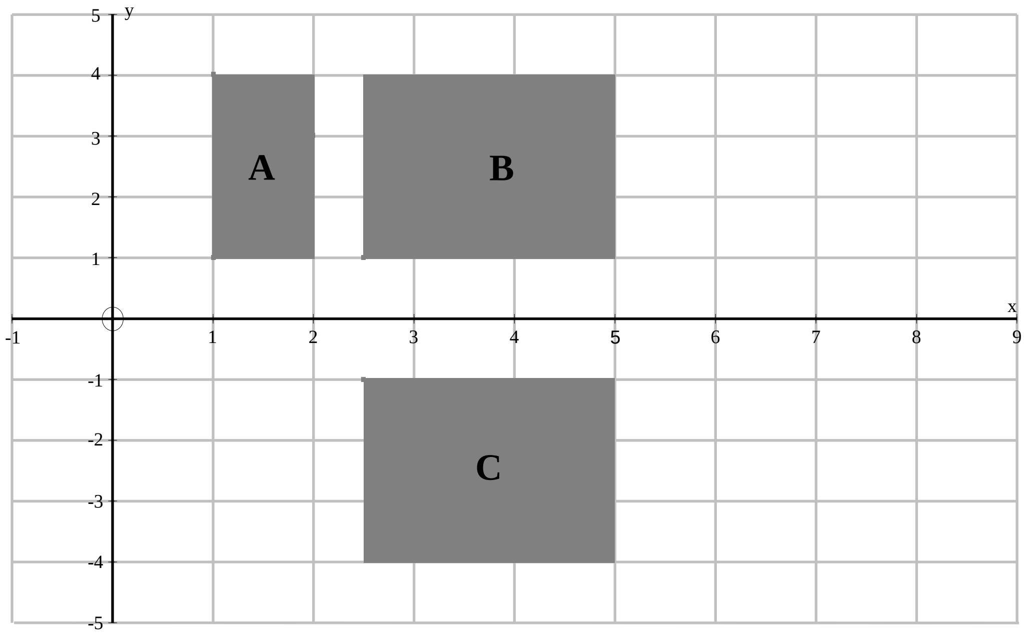
Will this work? Explain. (4)

15 marks

QUESTION 2

Describe the transformation and write down the single matrix which maps:

2.1 shape A onto shape B. (4)
2.2 shape B onto shape C. (2)
2.3 shape C onto shape A. (8)



14 marks

QUESTION 3

The determinant of the matrix, P, below is 20. Find the value of a given $a \in Z$

$$P = \begin{pmatrix} 1 & a & 1 \\ a & 4 & 3a \\ 3 & 1 & 2 \end{pmatrix}$$ (10)

10 marks

QUESTION 4

An AP Maths class is given the following system of equations:

$$2x + y - z = -7$$
$$x - y + z = 10$$
$$3x + 5y + 2z = 7$$

4.1 Mpho starts to find the solution by correctly applying row reduction but gives up. Complete her working and determine the solution to this system of equations.

Mpho's working:

Augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & -7 \\ 1 & -1 & 1 & 10 \\ 3 & 5 & 2 & 7 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 3 & -3 & -27 \\ 0 & 8 & -1 & -23 \end{array}\right)$$

(8)

4.2 Sandra decides to solve the equations by using the inverse of Matrix A where $A.X = Y$

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 3 & 5 & 2 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -7 \\ 10 \\ 7 \end{pmatrix}$$

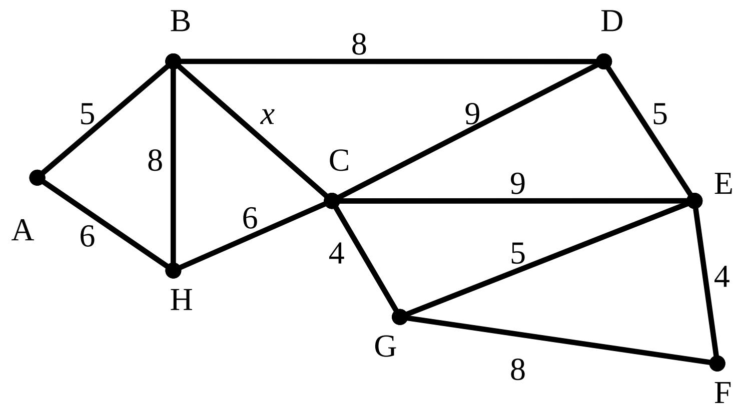
and $\det A = -21$.

(a) Find A^{-1}. (7)

(b) Express X in terms of A^{-1} and Y. (2)

17 marks

QUESTION 5



Referring to the above network answer the following questions:

5.1 Find the maximum value of x, where $x \in N$, that would ensure that edge BC is included in the unique shortest path between A and F. (10)

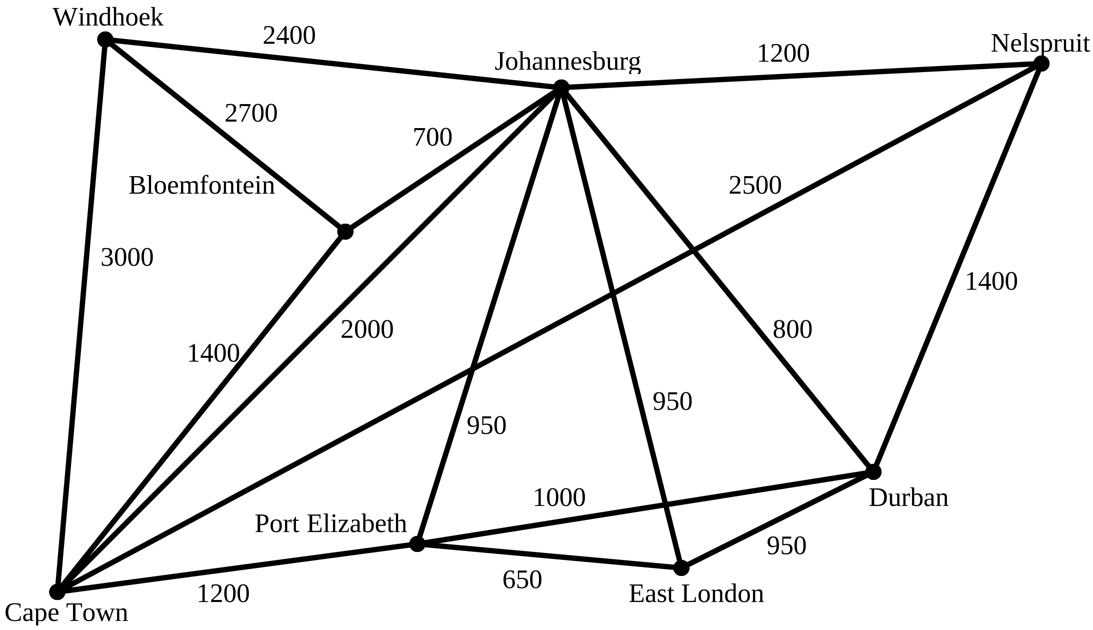
5.2 (a) List all the vertices of odd degree. (3)

(b) Which edges need to be duplicated in order to produce an Eulerian circuit of minimum weight? (7)

20 marks

QUESTION 6

Below is a diagram representing the cost of airfares between the major centres in South Africa and Namibia. A local businessman is based in Johannesburg and is required to visit each of the centres once a month.



6.1 Determine a lower bound for the cost of the airfares, by removing Johannesburg and using Kruskal's algorithm. Clearly state the order in which you choose the vertices. (9)

6.2 Giving reasons for your answers determine whether the lower bound found in 6.1 would be affected if:

(a) Prim's algorithm is used instead of Kruskal's algorithm. (2)

(b) Durban is used as the initial vertex instead of Johannesburg. (4)

6.3 Find an upper bound for the airfare cost using the nearest-neighbour algorithm and Johannesburg as the starting point. (9)

24 marks

Total for Module 4: 100 marks

Total: 300 marks