HILTON COLLEGE

FORM V

AUGUST 2010

ADVANCED MATHEMATICS

Time: $2\frac{1}{2}$ hours 120 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY.

1. This question paper consists of 8 pages and a **Formula Sheet**. Please check that your paper is complete.

2. Answer **ALL** the questions set on **FOUR** of the modules you studied.

3. Non-programmable calculators may be used, unless otherwise stated.

4. Please note that the diagrams are **NOT** necessarily drawn to scale.

Please do not turn over this page until you are asked to do so.

MODULE 1 INTEGRATION

QUESTION 1

a) Given: $\int_1^2 e^x dx = 4,67$ determine :

1) $\int_1^2 3e^x dx$ (1)

2) $\int_1^{1,7} e^x dx + \int_{1,7}^2 \left(e^x + 2\right) dx$ (4)

b) Given $\int_a^b \log_3 x \; dx = 5$, determine in terms of a and b $\int_a^b \log_3 9x \; dx$. (5)

[10]

QUESTION 2

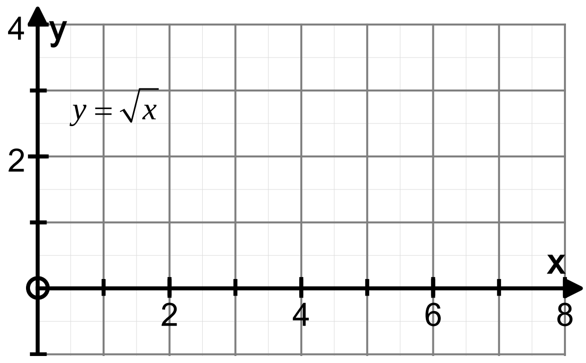
Using 6 subintervals, find two numbers m and M such that $m < \int_0^3 \left(x^2 + 2x - 2\right) dx < M$.

Hence give the approximate area between the parabola $y = x^2 + 2x - 2$ and the $x-$axis between $x = 0$ and $x = 3$.

[6]

QUESTION 3

The region bounded by the curve $y = \sqrt{x}$, the $x-$axis, $x = 0$ and $x = 4$ is shown in the diagram below:



3.1 Determine the area shown. (3)

3.2 Find the volume obtained by rotating the area shown through $360°$ about the x-axis. (4)

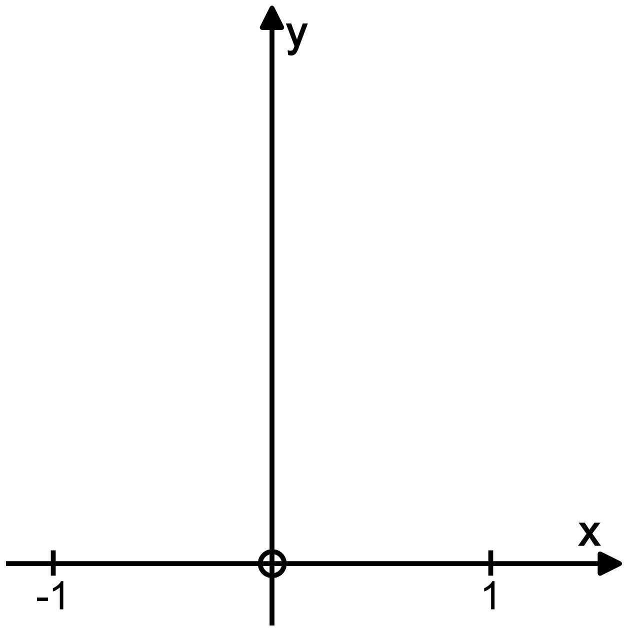
[7]

QUESTION 4

The functions f and g are defined on the domain $-1 \leq x \leq 1$ by

$f(x) = x^2$ and $g(x) = 2 - x^2$

Refer to the diagram below.



4.1 Describe the transformation that obtains the graph of g from the graph of f. (1)

4.2 Find the area of the region enclosed by the two curves. (3)

4.3 Show that the volume generated by rotating the region through $360°$ about the x-axis is $\frac{64}{15}\pi$. (4)

[8]

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MODULE 2 DIFFERENTIATION

QUESTION 1

a) Find

1) $\lim_{x \to 0} \frac{2 \sin x}{x}$ (1)

2) $\lim_{\theta \to 0} \frac{5\theta \cos \theta}{\sin 2\theta}$ (3)

b) Complete the following:

1) $\frac{d}{dx}(\sin x) = \ldots\ldots$ (1)

2) $\frac{d}{dx}(\cos x) = \ldots\ldots$ (1)

3) Assuming the derivatives in (1) and (2), prove that $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ (2)

Note: $\cot x = \frac{\cos x}{\sin x}$

c) Determine $\frac{dy}{dx}$ if:

1) $y = 2x\left(x^3 - 2\right)^{\frac{1}{3}}$ (3)

2) $y = \frac{\sqrt{x-1}}{\left(x^2 - 1\right)}$ (2)

(Leave your answer unsimplified.)

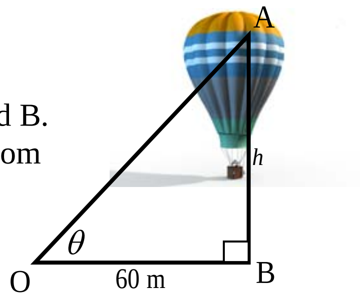
d) Differentiate $f(\theta) = \sin(\cos\theta)$ (2)

[15]

QUESTION 2

A hot-air balloon is rising vertically at a rate of 40 m/s above the ground B. The angle of elevation θ from a fixed point O of observation is 60 m from point B in the same horizontal plane as the balloon.

Determine the rate at which the angle θ is increasing when $\theta = 30°$.

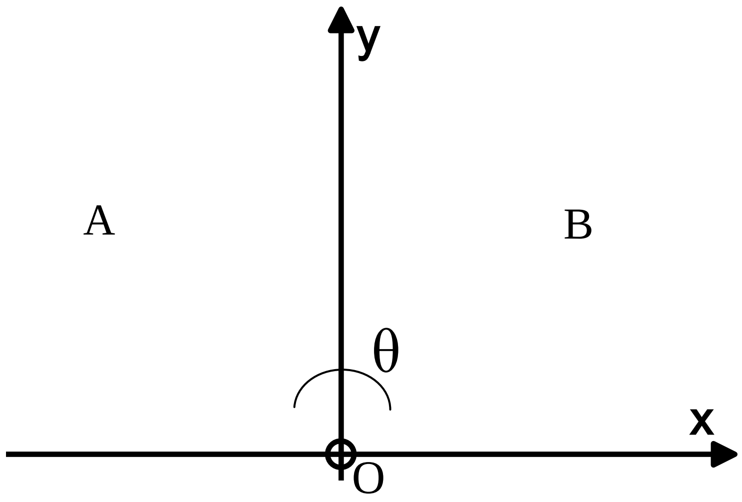


[6]

QUESTION 3

a) Find the equation of the tangent to the curve $y = \cos^2 2t + \cos 2t$ at the point on the curve whose t-value is $\frac{\pi}{4}$. (6)

b) A and B are points on the circumference of a semi-circle with equation $y = \sqrt{18 - x^2}$ such that $AB \parallel x-\text{axis}$. O is the origin.



1) If B has coordinates $\left(x; \sqrt{18 - x^2}\right)$, write down the coordinates of A. (1)

2) Write down the radius of the semi-circle. (1)

3) By using the formula Area of $\Delta ABC = \frac{1}{2} ab \sin\theta$, show that the area of the largest triangle AOB is 9 units^2. (4)

4) Hence, or otherwise, determine the size of the angle θ if the area of the segment AB (shaded part) is $(3\pi - 9)$ units^2 and $r^2 = 18$. (3)

[15]

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MODULE 4 MATRICES

QUESTION 1

1.1 Given any matrices A, B and C. Answer true or false:

1.1.1 $AB = BA$ (1)

1.1.2 $A(B+C) = AB + AC$ (1)

1.2 Find the matrix A if $AB = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. (3)

1.3 Suppose $B^2 = B + I$ where I is the 3×3 identity (unit) matrix and B is any 3×3 matrix. Show that

1.3.1 $(B+I)^2 = 3B + 2I$ (3)

1.3.2 $B^5 = 5B + 3I$ (2)

[10]

QUESTION 2

Find A^{-1} if $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$.

[8]

QUESTION 3

3.1 ΔOPQ has vertices $O(0;0)$, $P(3;0)$ and $Q(0;5)$. Find the matrix of the shear that takes ΔOPQ to ΔORQ, where R is the point with coordinates $(3;-9)$.
(3)

3.2 The transformation with matrix $\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{pmatrix}$ maps every point $(a;b)$ onto a particular straight line. Find the equation of the straight line. (3)

[6]

QUESTION 4

A system of 3 equations of the form $ax + by + cz = d$ are solved simultaneously and produces a solution with the following form:

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & k & m \end{array}\right)$$

Fig 1

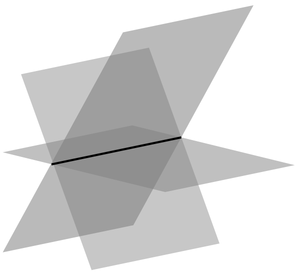


Fig 2

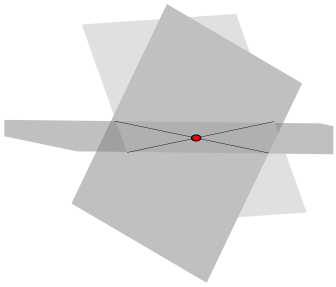
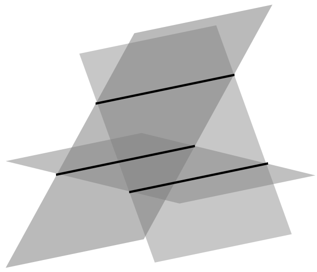


Fig 3



For which values of k and m will:

4.1 Figure 1 be a solution. (2)

4.2 Figure 2 be a solution. (2)

4.3 Figure 3 be a solution. (2)

[6]

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MODULE 7 PROBABILITY

QUESTION 1

(Note: $n! = n(n-1)(n-2)\ ...\ 2\times1$)

1.1 In how many ways can 5 boys and 4 girls be seated in a row so that the boys and girls are placed alternately? (3)

1.2 In how many ways can 6 people be arranged at a round table so that Geoff and Malusi do not sit together? (3)

1.3 Brother Benedict, a Medieval monk, had in his cell

2 books on geometry
2 books on music
3 books on arithmetic
4 books on astronomy

He kept his books on a single shelf so that books on the same subject were together. In how many ways could he have arranged his books on his bookshelf? (4)

1.4 How many different odd numbers can be made from the digits 2, 3, 4 and 5 if each digit is used once only? (5)

[15]

QUESTION 2

2.1 A bag contains 5 red and 6 green marbles. Two marbles are drawn at random from the bag. What is the probability of drawing marbles of different colour if the first marble is replaced before the second marble is drawn? (3)

2.2 E and F are events of a sample space S.
It is given that $\mathrm{P(E)} = 0,3$; $\mathrm{P(F)} = 0,6$ and $\mathrm{P(E \cap F)} = 0,1$.

How do we know
2.2.1 E and F are not independent events? (1)

2.2.1 E and F are not mutually exclusive events? (1)

Find
2.2.3 $\mathrm{P(\bar{E})}$ 2.2.4 $\mathrm{P(E \cup F)}$ 2.2.5 $\mathrm{P(E|F)}$ 2.2.5 $\mathrm{P(\bar{E}|F)}$ (6)

2.3 If 10% of the light bulbs produced by a factory are defective, what is the probability that out of 5 chosen at random, fewer than 2 will be defective? (4)

[15]

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