G.C. → SAINTS PRELIM 2010

1.1) $\sqrt[5]{(1+0{,}061)(1{,}071)(1{,}081)(1{,}091)(1{,}101)}$ → ans = 1,47551...

= 1,08090...

but $(1+i_{eff})^5 = (\ )(\ )(\ )$

∴ $1+i_{eff}$ = 1,0809...

∴ i_{eff} = 8,09%

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1.2) Pv = 225 000 $\quad P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$

n = 60

a) $i = \frac{0{,}125}{12}$ $\quad 225000 = x\left[\frac{1-(1+\frac{0{,}125}{12})^{-60}}{\frac{0{,}125}{12}}\right]$ → ans: 44,448...

∴ x = R 5062,04 (5062,036101) /6

b) $BO_{T_{30}}$: Note: Rounding errors since using ans from (a).

Pv at T_{30} → n = 30

$P_V = 5062{,}04\left[\frac{1-(1+\frac{0{,}125}{12})^{-30}}{\frac{0{,}125}{12}}\right]$

= 129847,81 (129847,8084)

or

BO = Loan + int − Fv (T_{30})

$= 225000(1+\frac{0{,}125}{12})^{30} - 5062{,}04\left[\frac{(1+\frac{0{,}125}{12})^{30}-1}{\frac{0{,}125}{12}}\right]$

= 129847,57 (..,5718)

Answer: No Rounding: R129847,7083..

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c) normal loan period = 60 months

After 30th payment → 30 pmts remain

After 12 months → 18 pmts remain

(method 1) -3 if missing

Loan amt:

$(129847{,}81 - 50\,000)(1+\frac{0{,}125}{12})^{12}$

= 90420,94132 → $= x\left[\frac{1-(1+\frac{0{,}125}{12})^{-18}}{\frac{0{,}125}{12}}\right]$

∴ x = 5535,08

inst:

(method 2)

$(129847{,}57 - 50000)(1+\frac{0{,}125}{12})^{12}$

= 90420,66954

$= x\left[\text{see alongside}\right]$

5535,06

Q2

x = R2000 ; $i = \frac{0{,}09}{12}$

2.1) Fv → n = 20 × 12 = 240

$F_V = 2000\left[\frac{(1+\frac{0{,}09}{12})^{240}-1}{0{,}09/12}\right]$ = 1335 773,74 /8

2.2) Int = total − sum of all pmts

= 1 335 773,74 − 240 (2000)

= 855 773,74 (..,7399) /4

2.3) BO_{T8} / n = 96 : → 5 pmts missed.

Lump Sum: (5 payments) 'grown' to year 10

∴ Lump sum: $2000\left(\frac{(1+\frac{0{,}09}{12})^5-1}{\frac{0{,}09}{12}}\right)\left(1+\frac{0{,}09}{12}\right)^{19}$ $\underbrace{19+5=24}_{2\text{ years}}$

= (10151,12923...) × (↓)

= 11699,58341.. = 11699,58 /12

2.4) 25% ans in (2.1) = 0,25 × 1335 773,74 = 333 943,435

$\underbrace{}$ 333 943,44

Allow 1001830,31

2.5) ∴ remaining amt: 1001 830,30 ← consistent rounding with prev answer.

For Perpetuity: $P_V = \frac{x}{i}$

∴ 1001 830,30 . i = x → monthly withdrawals.

$1001\,830{,}30 \times (\frac{0{,}12}{12})$ = 10 018,30 ← in perpetuity

Living annuity: 60 → 90 = 30 yrs ∴ n = 360

∴ $1001\,830{,}30 = x\left[\frac{1-(1+\frac{0{,}12}{12})^{-360}}{0{,}12/12}\right]$

∴ x = 10 304,95 /8

2.6) $x = 10\,000$ $\therefore n = ??$ (use logs)

$\therefore$ $1001\,830,30 = \overset{10\,000}{\downarrow}x\left[\dfrac{1-\left(1+\frac{0,12}{12}\right)^{-n}}{\frac{0,12}{12}}\right]$ and $\dfrac{0,12}{12} = 0,01$

$100,18303 = \left[\dfrac{1-(1,01)^{-n}}{0,01}\right]$

$0,0018303 = -(1,01)^{-n}$ (error) (no. soln)

Flawed → Since Perpetuity amt > 10 000.

Should be $x = 20\,000$

$1001\,830,30 = 20\,000\left[\dfrac{1-(1+0,01)^{-n}}{0,01}\right]$

$\therefore 0,5009\ldots = 1-(1,01)^{-n}$ /12

$\therefore 0,499\ldots = (1,01)^{-n}$

$\therefore \log_{1,01} 0,499\ldots = n = 69,84\ldots$ ie 70 withdrawals

$\therefore$ Last withdrawal: $1001\,830,30 = 20\,000\left[\dfrac{1-(1,01)^{-69}}{1,01}\right] + y(1,01)^{-70}$ /10

$\left[\underbrace{1001\,830,30 - 20\,000\left[\dfrac{1-(1,01)^{-69}}{0,01}\right]}_{8426\,310\ldots}\right](1,01)^{70} = y = \underline{16909,61}$,63 if used /31 in 2.5

,62 if no rounding for 75% amt /22

3).1) $T_2 = T_1 + 0,1\,T_1 = 1000 + 0,1 \times 1000 = 1100$

$T_3 = T_2 + 0,1\,T_2 = 1100 + 0,1 \times 1100 = 1210$

$T_4 = 1331$

$T_5 = 1464,1$

$A = 1000(1,1)^4 = 1464,1$

Geometric growth model = Compounding. /6

3.2 $108 = 50 + 1,2(50)\left[\dfrac{k-50}{k}\right]$

$\dfrac{29}{30} = \dfrac{k-50}{k}$ /6

$\therefore 29k = 30k - 1500$ $\therefore \underline{k = 1500}$

Calculus.

Qu.1) Prove: 9^n+3 div by 4

1) $n=1$: $9^1+3 = 12$ $\therefore$ div by 4 (True for $n=1$)

2) Assume true for $n=k$: $9^k+3 = 4p$ $(p \in N)$

3) prove true for $n=k+1$: $9^{k+1}+3 = 9^k.9+3$ but $9^k = 4p-3$

$\therefore (4p-3).9+3$

$= 36p - 27 + 3$

$= 36p - 24 = 4[9p-6]$

ie div by 4

$\therefore 9^n+3$ is div by 4 for all $n \in N$ /9

4.2) Continuous $\therefore \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$

for: $x<0$: $\lim\limits_{x\to 0^-}(x+1)^2 = 1$

for $x>0$: $\lim\limits_{x\to 0^+}\left(x-\frac{1}{2}k\right)^2 = \frac{1}{4}k^2$

$\therefore 2p+k = 1$ → $\frac{1}{4}k^2 = 1$

$k^2 = 4$

$\therefore k = \pm 2$

$\therefore 2p+2 = 1$ or $2p-2 = 1$

$2p = -1$ | $2p = 3$

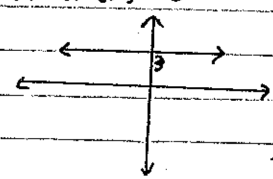
$p = -\frac{1}{2}$ | $p = \frac{3}{2}$

$\therefore k = 2$; $p = -\frac{1}{2}$

or $k = -2$; $p = \frac{3}{2}$ /1

4.3) $h'(x) = 3x - 6$ (str line eqn)

a) $\therefore h''(x) = 3$



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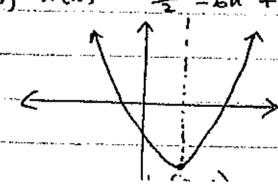
b) $h(x) = \frac{3x^2}{2} - 6x + c$



From orig sketch "$y=0$ at $x=2$" $\therefore x = 2$ a.o. /4

4.4) $A = \frac{1}{2}r^2\theta$ and $s = r\theta$

$\therefore 72 = \frac{1}{2}r^2\theta$

$\therefore \theta = \frac{144}{r^2}$

Perim $= 2r + r\theta = 36$

$$2r + r\left(\frac{144}{r^2}\right) = 36$$

$$2r + \frac{144}{r} = 36 \quad (\times r)$$

$$2r^2 + 144 - 36r = 0$$

$$r^2 - 18r + 72 = 0$$

$$(r-12)(r-6) = 0$$

$$\therefore r = 12 \text{ or } r = 6$$

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5.1) a) $\lim_{x\to 3} \frac{2\sqrt{x} - \sqrt{12}}{3-x}$ $\sqrt{4 \cdot 3}$ $= \lim_{x\to 3} \frac{2(\sqrt{x} - \sqrt{3})}{(\sqrt{3} - \sqrt{x})(\sqrt{3}+\sqrt{x})}$

$$= \lim_{x\to 3} \frac{-2}{\sqrt{3}+\sqrt{x}} = \frac{-2}{\sqrt{3}+\sqrt{3}}$$

$$= \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$ /8

b) $\lim_{x\to 0} \frac{5\cancel{\sin 2x}}{2\cancel{\sin 2x}\cos 2x} = \lim_{x\to 0} \frac{5}{2\cos 2x} = \frac{5}{2}$ /6

c) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2(1-\frac{1}{x})}} = \lim_{x\to\infty} \frac{x}{x\sqrt{1-\frac{1}{x}}}$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x}}} = \frac{1}{\sqrt{1-0}} = 1$$ /8

5.2) LHS: $\frac{1 + \frac{1}{\cos 2A}}{\cos^2 A}$ RHS: $\frac{2}{\cos 2A}$

Leave $= \frac{\frac{\cos 2A + 1}{\cos 2A}}{\cos^2 A}$ $\Big\}$ $\cos 2A = 2\cos^2 A - 1$

$$= \frac{(2\cos^2 A - 1) + 1}{\cos 2A} \times \frac{1}{\cos^2 A}$$

$$= \frac{2\cancel{\cos^2 A}}{\cos 2A} \times \frac{1}{\cancel{\cos^2 A}} = \frac{2}{\cos 2A} = \text{RHS} \quad \text{QED}$$ /10

6) 1) $y = (\tan 2x)^{\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(\tan 2x)^{-\frac{1}{2}} \cdot \sec^2(2x) \cdot 2$$

$$= \frac{1}{\sqrt{\tan 2x}} \cdot \frac{1}{\cos^2 2x}$$

$x = \frac{\pi}{8} \implies 45°$

$\therefore$ at $x = \frac{\pi}{8}$

$$= \frac{1}{\sqrt{\tan\frac{\pi}{4}}} \cdot \frac{1}{\cos^2(\frac{\pi}{4})}$$

equal to $\tan 45° = 1$

$$= 1 \cdot \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$= \frac{1}{\frac{2}{4}} = \frac{1}{\frac{1}{2}} = 2$$ /9

6.2) implicit: $y^3 + 3xy^2 \frac{dy}{dx} + 6x = \cancel{4}y + x\frac{dy}{dx}$

$$\therefore 3xy^2 \frac{dy}{dx} - x\frac{dy}{dx} = y - 6x - y^3$$

$$\frac{dy}{dx} = \frac{y - 6x - y^3}{3xy^2 - x}$$

at $(-2;1) \implies m = \frac{1 - 6(-2) - 1}{3(-2)(1) - (-2)} = \frac{12}{-4} = -3$ /1

6.3) $f(x) = (x-2)^{-1}$ $\therefore f'(x) = -1(x-2)^{-2}$

$$f''(x) = 2(x-2)^{-3}$$

$$f'''(x) = -6(x-2)^{-4}$$

$$f^{(4)}(x) = 24(x-2)^{-5}$$

$$\therefore f^{(n)}(x) = (-1)^n \cdot n! \cdot (x-2)^{-(n+1)}$$ /7

7.1) a) $\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$

$\therefore \frac{d^2y}{dx^2} = \frac{(\cos x - x\sin x - \cos x)x^2 - 2x(x\cos x - \sin x)}{x^4}$

$= \frac{x^2\cos x - x^3\sin x - x^2\cos x - 2x^2\cos x + 2x\sin x}{x^4}$

Can still cancel out x

$= \frac{-x^3\sin x - 2x^2\cos x + 2x\sin x}{x^4}$

Easier approach:

$\frac{\cos x}{x} - \frac{\sin x}{x^2}$

$\Rightarrow \frac{-x\sin x - \cos x}{x^2} - \left[\frac{x^2\cos x - 2x\sin x}{x^4}\right]$

$= \frac{-x\sin x - \cos x}{x^2} + \frac{-x^2\cos x + 2x\sin x}{x^4}$

$\frac{4}{10}$ for 1st deriv only. /10

b) $y = (x-1)^{\frac{1}{2}} \cdot \tan x$

$\therefore \frac{dy}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}} \cdot \tan x + (x-1)^{\frac{1}{2}} \cdot \sec^2 x$

$= \frac{\tan x}{2\sqrt{x-1}} + \sqrt{x-1} \cdot \sec^2 x$ /10

7.2) (a) $f(x) = 2\cos\theta + \theta - 2 = 0$

$f(3) = -0,9799\ldots$; $f(5) = 3,567\ldots$

$\underbrace{}_{\text{change in sign}}$ $\therefore$ solution interval $\theta \in [3;5]$ /1

(b) let $x_1 = 4$ (any value within $\theta \in [3;5]$)

$f'(x) = -2\sin\theta + 1$

$\therefore x_{r+1} = x_r^{\theta} - \frac{(2\cos\theta + \theta - 2)}{-2\sin\theta + 1}$ ✓ let $\theta = \cancel{34}$ ans

$\therefore x_2 = 3,7244\ldots$

$x_3 = 3,6984\ldots$

$\therefore \theta = 3,69815$ (5 d.p.) (3,698153673)

8) 8.1) $f'(x) = 15x^4 - 15x^2 = 0$

$\therefore x^4 - x^2 = 0$

$x^2(x^2 - 1) = 0$

$\therefore x = 0$ or $x = \pm 1$

✓ using knowledge of inequalities
if $f'(x) > 0$ increasing
$f'(x) < 0$ decreasing

+   -1   -   0   -   1   +

$\therefore$ increasing $(-\infty;-1)$; $(1;\infty)$
decreasing $(-1;0)$; $(0;1)$ /7

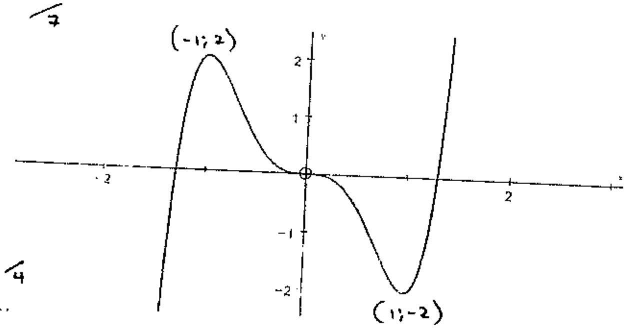
8.2) Intercepts: $\therefore f(x) = 0$

$x^3(3x^2 - 5) = 0$

$\therefore x = 0$ or $x^2 = 5/3$

$\therefore x = \pm\sqrt{5/3}$ /4

$\therefore x = \pm 1,290\ldots$



8.3) t pts: $f'(x) = 0$ $\therefore$ $x = 0$; $x = 1$; $x = -1$

$f(0) = 0$; $f(1) = -2$; $f(-1) = 2$

$f''(x) = 60x^3 - 30x$; $f''(1) = 30$ (min) ; $f''(-1) = -30$ (max) /7

$\therefore f''(0) = 0$ ← from (a) this is pt of inflection

9) 1) $f'(x) = \frac{2(3x-1) - 3(2x+1)}{(3x+1)^2}$

$= \frac{6x - 2 - 6x - 3}{(3x+1)^2} = \frac{-5}{(3x+1)^2}$

But t pt: $f'(x) = 0$ $\quad \frac{u}{v} \rightarrow \frac{u'v - v'u}{v^2}$

$\therefore \frac{-5}{(3x+1)^2} = 0$

but no x value will yield $f'(x) = 0$

$\therefore$ no tpt's. /5

9.2) Ass: vertical Ass: $x = \frac{1}{3}$

horiz Ass:

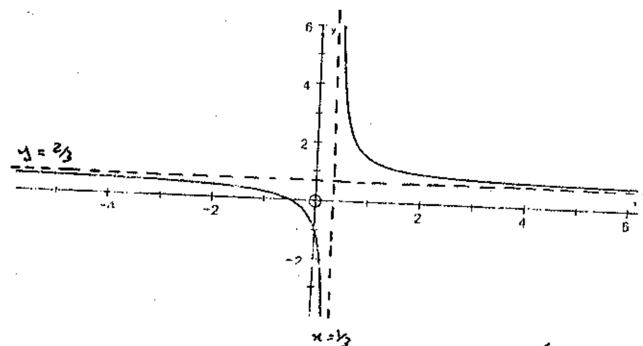
$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2 + 1/x}{3 - 1/x} = \frac{2}{3}$

y int: $f(0) = \frac{1}{-1} = -1$

x int: $f(x) = 0$

$\therefore 2x + 1 = 0$

$\therefore x = -\frac{1}{2}$



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Since $x > 0$

Q10) Use Riemann Sum $\rightarrow \int_0^2 (-x^2+4)\,dx$

$A = \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} f(x_i)$

Partition width $= \frac{2}{n}$

$\therefore x_i = a + (\text{width})i$
$= \left(0 + \frac{2i}{n}\right)$

$\therefore f(x_i) = -\left[\quad + \frac{2i}{n}\right]^2 + 4$
$= -\left[\frac{4i^2}{n^2}\right] + 4 = \frac{-4i^2}{n^2} + 4$

$\therefore A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\frac{-4i^2}{n^2} + 4\right)$

$= \lim_{n\to\infty} \left(\frac{-8}{n^3} \sum_{i=1}^{n} i^2 + \frac{8}{n} \sum_{i=1}^{n} 1\right)$

$= \lim_{n\to\infty} \left(\frac{-8}{n^3}\left[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right] + \frac{8}{n}[n]\right)$

$= \lim_{n\to\infty} \left[-8/3 - 4/n - 4/3n + 8\right]$

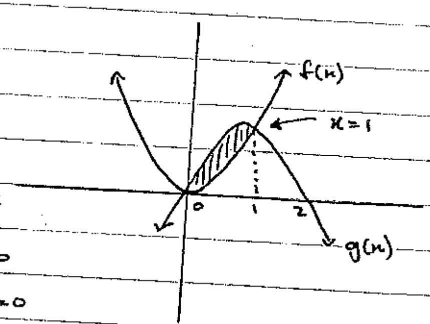
$= \left[-8/3 + 8\right]$ /10

$= 16/3$ units^2

10.2) $f(x) = x^2$

$g(x) = -x^2 + 2x$

x ints: $-x^2+2x = 0$
$x(-x+2) = 0$
$x = 0$ or 2



Intersection: $x^2 = -x^2 + 2x$
$2x^2 - 2x = 0$
$x^2 - x = 0$
$\therefore x(x-1) = 0$

$\therefore$ Area $= \int_0^1 \underset{\text{top - bottom}}{g(x) - f(x)}$

$= \int_0^1 \left[(-x^2+2x) - x^2\right] dx$

$= \int_0^1 (-2x^2+2x)\,dx$

$= \left[-\frac{2x^3}{3} + x^2\right]_0^1$ /9

$= -2/3 + 1 = \boxed{1/3}$ units^2

11).1) $\int x^2\sqrt{x^3+1}\,dx$

let $x^3+1 = u$
$3x^2\,dx = du$
$dx = du/3x^2$

$= \int x^2 \sqrt{u}\,\frac{du}{3x^2}$

$= 1/3 \int u^{1/2}\,du$

$= 1/3 \frac{u^{3/2}}{3/2} + c$

$= \frac{2}{9}u^{3/2} + c = \frac{2}{9}(x^3+1)^{3/2} + c$ /9

11.2) $\int \text{cosec}\,x \cdot \cot x\,dx = -\int -\text{cosec}\,x \cot x\,dx$

$= -\text{Cosec}\,x + c$ /4