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St Stithians Boys College FP Maths Prelim 2009

SECTION A

Question 1

1.1. $|3x^2-2x| = 1$

$3x^2-2x-1=0$ OR $-(3x^2-2x)-1=0$

$\therefore (x-1)(3x+1)=0$ $3x^2-2x+1=0$

$\therefore x=1$ OR $x=-\frac{1}{3}$ $\therefore$ no real solutions

1.2. $3e^x - \frac{2}{e^x} = 1$

let $e^x = k$

$\therefore 3k - \frac{2}{k} = 1$

$\therefore 3k^2 - k - 2 = 0$

$\therefore (k-1)(3k+2)=0$

$\therefore e^x = 1$ OR $e^x = -\frac{2}{3}$

$\therefore x = \ln 1$ OR $x \neq \ln(-\frac{2}{3})$

$\therefore x = 0$

1.3. $\ln(e^{2x}-12) = x$

$\therefore e^x = e^{2x} - 12$

$\therefore e^{2x} - e^x - 12 = 0$

$\therefore (e^x-4)(e^x+3)=0$

$\therefore e^x = 4$ OR $e^x \neq -3$

$\therefore x = \ln 4$

$\therefore x = 1{,}39$

1.4 $\frac{8}{x+6} \leq 2$

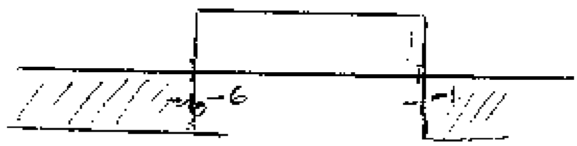
$\therefore \frac{8}{x+6} - 2 \leq 0$

$\therefore \frac{8-2(x+6)}{x+6} \leq 0$

$\therefore \frac{-2x+2}{x+6} \leq 0$

$\therefore \frac{-(x+1)}{x+6} \leq 0$

$\therefore$ CVs $x=-1$; $x=-6$



$\therefore x \leq -6$; $x \geq -1$

<u>Question 2</u>

2.1 $\frac{(1-4i)^2}{i+1}$

$= \frac{1-8i+16(-1)}{i+1}$

$= \frac{-15-8i}{i+1} \times \frac{(i-1)}{(i-1)}$

$= \frac{-15i+15+8+8i}{-1-1}$

$= \frac{-7i+23}{-2}$

$= \frac{7i}{2} - \frac{23}{2}$

$= \frac{-23}{2} + \frac{7}{2}i$

2.2. $x^3+x^2+4x+30=0$

$x=1\pm 3i$

$\therefore (x-1+3i)(x-1-3i)$

$\therefore (x^2-x-\cancel{3xi}-x+1+\cancel{3i}+\cancel{3xi}-\cancel{3i}+9)$

$\therefore (x^2-2x+10)(x+3)=0$

$\therefore \underline{x=1\pm 3i \quad \text{or } x=-3}$

2.3. i^{3n+2} is n a real number if $0\leq n\leq 155$

$i^1=i \qquad n=0 \quad i^2=-1 \quad \therefore$ real

$i^2=-1 \qquad n=1 \quad i^{3+2}=i^5=i$

$i^3=-i \qquad n=2 \quad i^{3(2)+2}=i^8=1 \quad$ real

$i^4=1 \qquad n=3 \quad i^{3(3)+2}=i^{14}=-i$

$n=4 \quad i^{3(4)+2}=i^{14}=-1 \quad$ real

$\therefore$ every second solution is real

$\therefore$ number of solutions $=\dfrac{155+1}{2}$

$=\underline{78}$

Question 3

$$f(x)=\begin{cases}\dfrac{a}{x} & x\geq 1\\ b-2x & x<1\end{cases}$$

$f(x)$ is differentiable at $x=1$ $\rightarrow$ $\therefore \lim_{x\to 1^-} f'(x)=\lim_{x\to 1^+} f'(x)$

$\therefore f(x)$ is also continuous at $x=1$

$\therefore \lim_{x\to 1^-} f(x)=f(1)=\lim_{x\to 1^+} f(x)$

$\therefore a=b-2 \quad \ldots (1)$

$\therefore \lim_{x\to 1^-} -a.x^{-2}=\lim_{x\to 1} -2$

$\therefore -a=-2 \quad (2)$

$\therefore a=2$

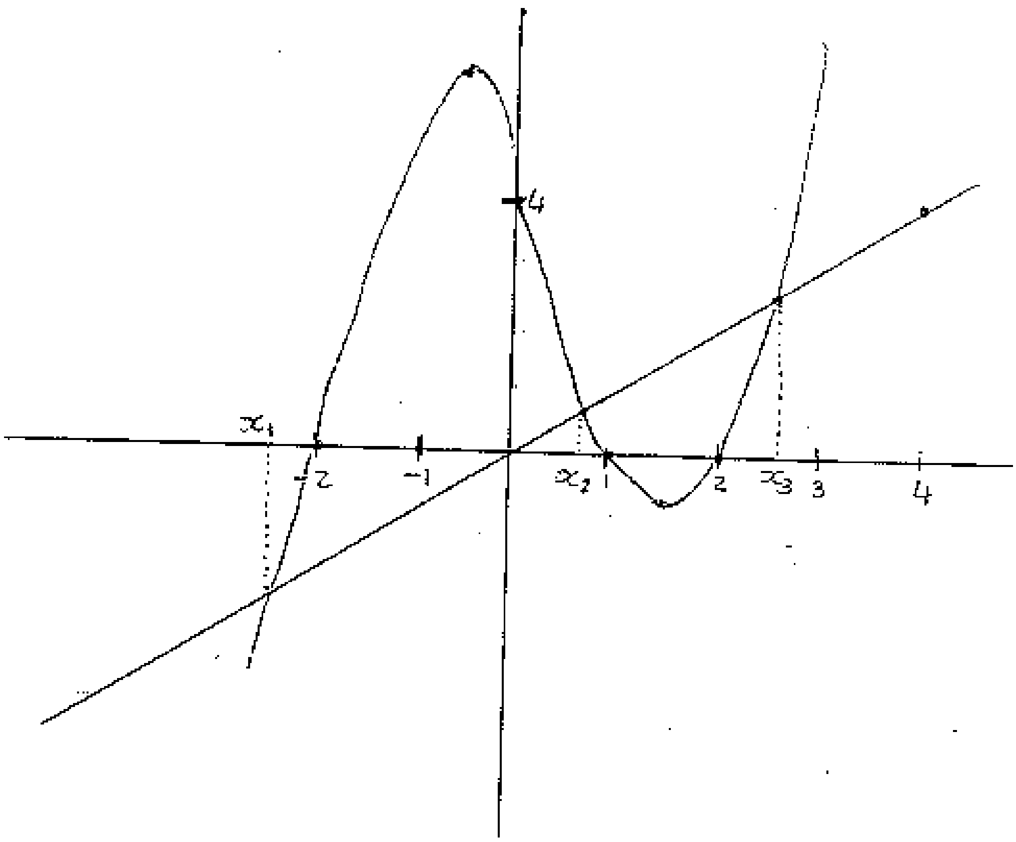
$\therefore 2=b-2$

$\therefore b=4$

$\therefore \underline{a=2\ ;\ b=4}$

Question 4

$y = x^3 - x^2 - 4x + 4 \quad = (x-2)(x+2)(x-1)$



$\frac{dy}{dx} = 3x^2 - 2x - 4$

$0 = 3x^2 - 2x - 4$

$\therefore x = 1{,}54 \quad x = -0{,}89$

$f(1{,}54) = -0{,}88$

$f(-0{,}89) = 6{,}06$

Newtons method   to solve $x^3 - x^2 - 4x + 4 = x$

$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$

let $x_0 = 2 = \text{ans}$

$\therefore x_{r+1} = \text{ans} - \frac{\text{ans}^3 - \text{ans}^2 - 5\text{ans} + 4}{3\text{ans}^2 - 2\text{ans} - 4}$

$x_1 = 2{,}5$

$x_2 = 2{,}41025641$

$x_3 = 2{,}393832209$

$x_4 = 2{,}3931678314$

$x_5 = 2{,}391417772$

$x_6 = 2{,}391386608$

$x_7 = 2{,}391382855$

$x_8 = 2{,}391382441$

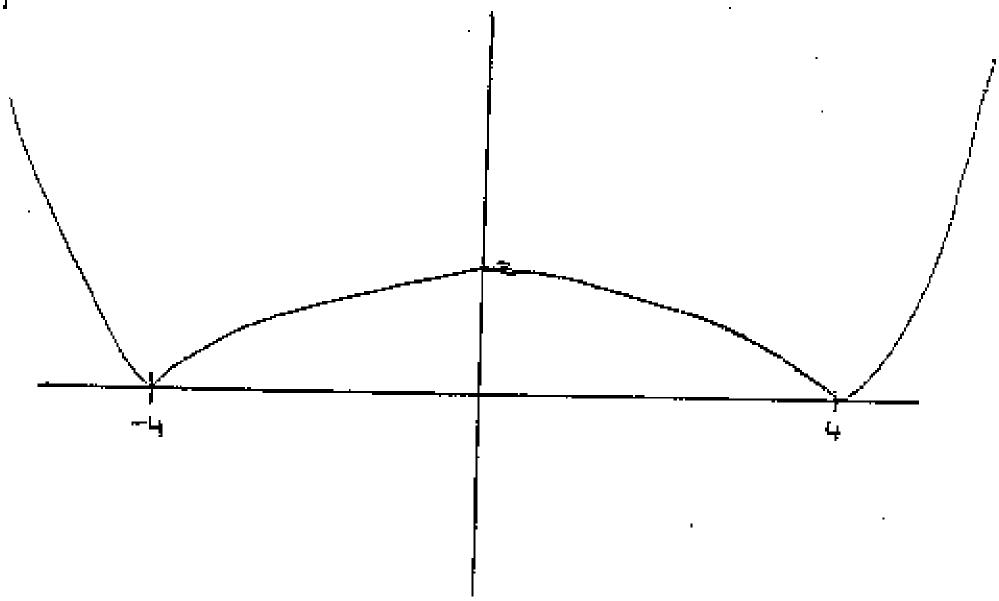
$x_9 = 2{,}391382358$

$x_{10} = 2{,}391382351$

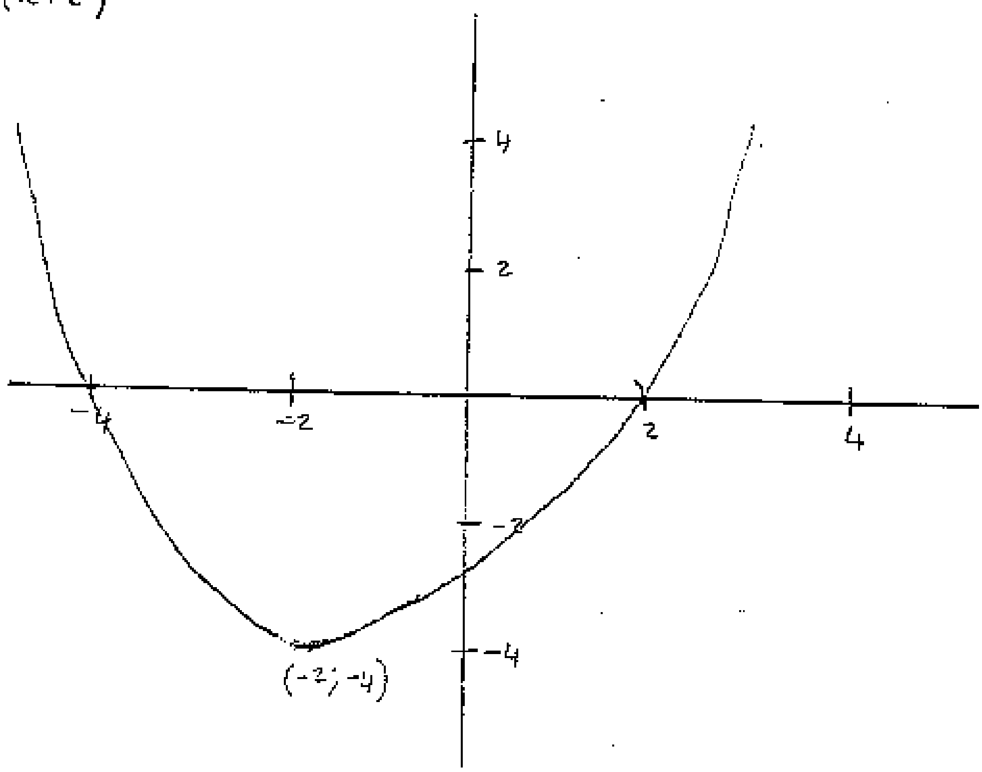
$\therefore x = 2{,}391$

Question 5

5.1. $y = |f(x)|$



5.2. $y = 2 \times f(x+2)$



Question 6

6.1.1. $\lim\limits_{x \to \infty} \dfrac{5 - 3x - 2x^2}{(2x-3)^2}$

$= \lim\limits_{x \to \infty} \dfrac{-2x^2 - 3x + 5}{4x^2 - 12x + 9} \times \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim\limits_{x \to \infty} \dfrac{-2 - \frac{3}{x} + \frac{5}{x^2}}{4 - \frac{12}{x} + \frac{9}{x^2}}$

$= -\dfrac{1}{2}$

6.1.2. $\lim_{x \to 0} \frac{x^2}{\tan 2x \tan 3x}$

$\lim_{x \to 0} x^2 \times \frac{\cos 2x \cos 3x}{\sin 2x \sin 3x}$

$= \lim_{x \to 0} \frac{x}{\sin 2x} \cdot \frac{x}{\sin 3x} \cdot \cos 2x \cos 3x$

$= \lim_{x \to 0} \frac{2x}{\sin 2x} \cdot \frac{3x}{\sin 3x} \cdot \cos 2x \cos 3x \cdot \frac{1}{6}$

$= 1 \times 1 \times 1 \times 1 \times \frac{1}{6}$

$= \frac{1}{6}$

6.1.3 $\lim_{x \to 1} \frac{3^{2x} - 9}{3^x - 3}$

$= \lim_{x \to 1} \frac{(3^x + 3)\cancel{(3^x - 3)}}{\cancel{(3^x - 3)}}$

$= \lim_{x \to 1} 3^x + 3$

$= 6$

6.2. $f(x) = \frac{-3}{x^2}$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\frac{-3}{(x+h)^2} - \frac{-3}{x^2}}{h}$

$= \lim_{h \to 0} \frac{-3x^2 + 3(x^2 + 2xh + h^2)}{h(x^4 + 2x^3h + x^2h^2)}$

$= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}(x^4 + 2x^3h + x^2h^2)}$

$= \frac{6x}{x^4}$

$= \frac{6}{x^3}$

6.3.1 $y = \frac{5}{x}\left(1-\sqrt{x}\right)$

$y = 5 \cdot x^{-1}\left(1 - x^{\frac{1}{2}}\right)$

$\therefore y = 5x^{-1} - 5x^{-\frac{1}{2}}$

$\therefore \frac{dy}{dx} = -5x^{-2} + \frac{5}{2}x^{-\frac{3}{2}}$

$\frac{dy}{dx} = \frac{-5}{x^2} + \frac{5}{2x^{\frac{3}{2}}}$

6.3.2 $x^3 + y^3 = xy$ $\qquad (uv)' = u'v + uv'$

$3x^2 + 3y^2\frac{dy}{dx} = y + x\frac{dy}{dx}$

$\therefore 3y^2\frac{dy}{dx} - x\frac{dy}{dx} = y - 3x^2$

$\frac{dy}{dx}\left(3y^2 - x\right) = y - 3x^2$

$\frac{dy}{dx} = \frac{y-3x^2}{3y^2-x}$

6.3.3 $y = \sqrt{x^2+1} + 4(3x+1)^5$

$y = (x^2+1)^{\frac{1}{2}} + 4(3x+1)^5$

$\therefore \frac{dy}{dx} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x + 20(3x+1)^4 \cdot 3$

$\frac{dy}{dx} = \frac{x}{(x^2+1)^{\frac{1}{2}}} + 60(3x+1)^4$

6.3.4 $y = \sqrt{x}\cos^3 2x$ $\qquad (uv)' = u'v + uv'$

$y = x^{\frac{1}{2}}(\cos 2x)^3$

$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(\cos 2x)^3 + x^{\frac{1}{2}}\left(3(\cos 2x)^2 \cdot -\sin 2x \cdot 2\right)$

$\therefore \frac{dy}{dx} = \frac{\cos^3 2x}{2x^{\frac{1}{2}}} - 6x^{\frac{1}{2}}\cos^2 2x \sin 2x$

64 $f(x) = \frac{1}{2x^2}$

$f(x) = \frac{1}{2} x^{-2}$

$\therefore f'(x) = \frac{1}{2} \cdot -2\, x^{-3}$

$f''(x) = \frac{1}{2} \cdot -2 \cdot -3\, x^{-4}$

$f^{(3)}(x) = \frac{1}{2} \cdot -2 \cdot -3 \cdot -4\, x^{-5}$

$\therefore f^{(n)}(x) = \frac{1}{2} \times (-1)^n \cdot (n+1)! \cdot x^{-(n+2)}$

$\therefore f^{(n)}(x) = \frac{\frac{1}{2} \times (-1)^n \cdot (n+1)!}{x^{n+2}}$

$\therefore f^{(n)}(x) = \frac{(-1)^n \times (n+1)!}{2x^{n+2}}$

6.5. $y = 2\tan^2\left(\frac{\pi}{2} - 3\theta\right)$; $\theta = \frac{\pi}{4}$

$y = 2\left(\tan\left(\frac{\pi}{2} - 3\theta\right)\right)^2$

$\frac{dy}{dx} = 4\left(\tan\left(\frac{\pi}{2} - 3\theta\right)\right) \cdot \sec^2\left(\frac{\pi}{2} - 3\theta\right) \cdot -3$

$\therefore \frac{dy}{dx} = -12\tan\left(\frac{\pi}{2} - 3\theta\right) \cdot \frac{1}{\cos^2\left(\frac{\pi}{2} - 3\theta\right)}$

$\therefore \frac{dy}{dx} = \frac{-12\sin\left(\frac{\pi}{2} - 3\theta\right)}{\cos^3\left(\frac{\pi}{2} - 3\theta\right)}$

$\therefore$ at $\theta = \frac{\pi}{4}$ $\frac{dy}{dx} = \frac{-12\sin\left(\frac{\pi}{2} - 3\left(\frac{\pi}{4}\right)\right)}{\cos^3\left(\frac{\pi}{2} - 3\left(\frac{\pi}{4}\right)\right)}$

$= 24$

Question 7

$$h(x) = \frac{2x^3}{x^2-4}$$

7.1. domain of $h(x)$

$x \in \mathbb{R}$ but $x \neq \pm 2$

7.2. y intercept $h(0) = 0$ $\therefore (0;0)$

x intercept $0 = 2x^3$

$\therefore x = 0$

7.3. Stationary points

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$h'(x) = \frac{6x^2(x^2-4) - 2x^3(2x)}{(x^2-4)^2}$$

$$= \frac{6x^4 - 24x^2 - 4x^4}{(x^2-4)^2}$$

$$= \frac{2x^4 - 24x^2}{(x^2-4)^2}$$

$\therefore 0 = 2x^4 - 24x^2$

$\therefore 0 = x^2(x^2 - 12)$

$\therefore x = 0$ OR $x = \pm\sqrt{12}$

$\therefore x = 0$ OR $x = \pm 3{,}46$

$f(0) = 0$ $\quad f(\sqrt{12}) = \frac{2(\pm\sqrt{12})^3}{(\pm\sqrt{12})^2 - 4} = \pm 6\sqrt{3}$

$\therefore (0;0)$; $(2\sqrt{3}; 6\sqrt{3})$; $(-2\sqrt{3}; -6\sqrt{3})$

To classify $f'(x)$: $+$ | $-2\sqrt{3}$ | $-$ | 0 | $-$ | $2\sqrt{3}$ | $+$

$\therefore (0;0) \rightarrow$ point of inflectron

$(2\sqrt{3}; 6\sqrt{3}) \rightarrow$ local maximum

$(-2\sqrt{3}; -6\sqrt{3}) \rightarrow$ local minimum

7.4. vertical asymptotes → where undefined

$\therefore (x^2 - 4) = 0$

$\therefore x^2 = 4$

$\therefore x = \pm 2$

Oblique asymptote

$$m = \lim_{x \to \infty} \frac{f(x)}{x}$$

$$= \lim_{x \to \infty} \frac{2x^3}{x(x^2-4)}$$

$$= \lim_{x \to \infty} \frac{2x^3}{x^3 - 4x} \times \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{2}{1 - \frac{4}{x^2}}$$

$$= 2$$

$$c = \lim_{x \to \infty} f(x) - mx$$

$$= \lim_{x \to \infty} \frac{2x^3}{x^2-4} - 2x$$

$$= \lim_{x \to \infty} \frac{2x^3 - 2x(x^2-4)}{x^2-4}$$

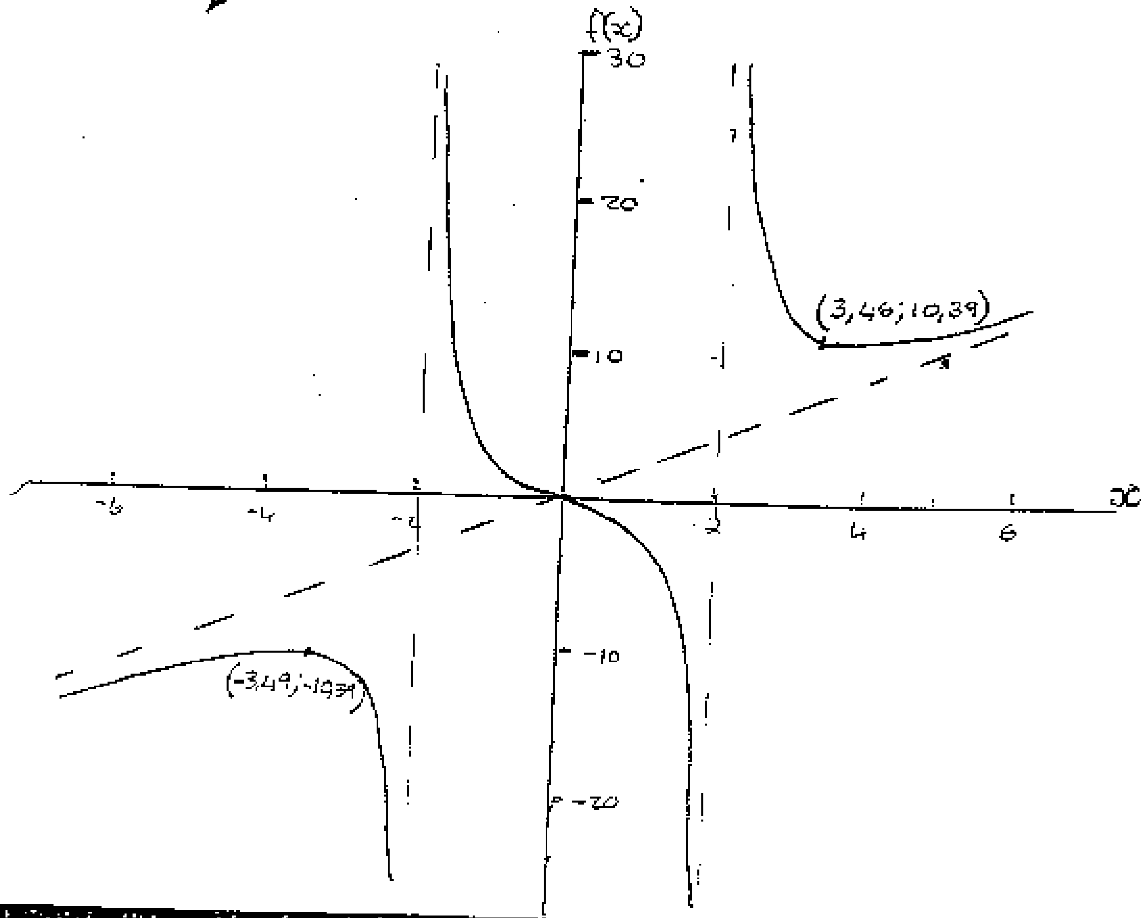
$$= \lim_{x \to \infty} \frac{4}{x^2-4} \times \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{\frac{4}{x^2}}{1 - \frac{4}{x^2}}$$

$$= 0$$

$\therefore y = 2x$

7.5



Question 8

8.1. Area $= \frac{1}{2} r^2 \theta$

$\therefore$ Area shaded $= \frac{1}{2} \times 5^2 \times \frac{5}{9}\pi - \frac{1}{2} \times 3^2 \times \frac{5}{9}\pi$

$= \frac{40}{9}\pi$ units^2

8.2. Area ACB $= \left(\frac{1}{2} \times 5^2 \times \sin\frac{5\pi}{9} - \frac{1}{2} 3^2 \times \frac{5}{9}\pi\right) \times \frac{1}{2}$

$= 2{,}23$ units^2

Question 9

Prove $\log x + 2\log x + 3\log x + \ldots + n\log x = \frac{n}{2} \log x^{n+1}$

1) Prove true for $n=1$

LHS $1\log x$

RHS: $\frac{1}{2} \log x^2$

$= 2 \cdot \frac{1}{2} \log x$

$= \log x$

$\therefore$ RHS = LHS

$\therefore$ true for $n=1$

2) Assume true for $n=k$

$\therefore \log x + 2\log x + 3\log x + \ldots + k\log x = \frac{k}{2} \log x^{k+1}$

3) Prove true for $n=k+1$

$\therefore \log x + 2\log x + 3\log x + \ldots + k\log x + (k+1)\log x = \frac{k}{2}\log x^{k+1} + (k+1)\log x$

Target $\rightarrow \frac{k+1}{2} \log x^{k+2} = \frac{(k+1)(k+2)}{2} \log x$

RHS: $\frac{k}{2}\log x^{k+1} + (k+1)\log x$

$= \frac{k(k+1)}{2} \log x + (k+1)\log x$

$= (k+1)\log x\left(\frac{k}{2}+1\right)$

$= (k+1)\left(\frac{k}{2}+1\right)\log x$

$= \frac{k^2+3k+1}{2} \log x$

$= \frac{(k+1)(k+2)}{2} \log x$

$=$ target $\therefore$ true for $n=k+1$ $\therefore$ true for all n

SECTION B

Finance and Modelling

Question 1

$8; 6; b; -66; -142$

$T_n = aT_{n-1} - 4T_{n-2}$

1) $b = a(6) - 4(8)$

$\therefore b = 6a - 32$

2) $-142 = a(-66) - 4b$

$\therefore -142 = -66a - 4b$

$\therefore -142 = -66a - 4(6a - 32)$

$-142 = -66a - 24a + 128$

$\therefore 90a = 270$

$\therefore a = 3$

$\therefore b = 6(3) - 32$

$b = -14$

Question 2

2.1. $4 = (1+i)^8$

$\therefore i = \sqrt[8]{4} - 1$

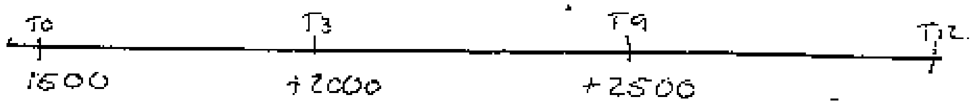
$\therefore i = 18,92\%$ p.a.

2.2 $4 = \left(1 + \frac{i}{4}\right)^{32}$

$\therefore i = \left(\sqrt[32]{4} - 1\right) \times 4$

$\therefore i = 17,71\%$ p.a.c.q

Question 3



$i = 9{,}5\%$ p.a.c.m.

3.1. $A = 1500\left(1+\frac{0{,}095}{12}\right)^{144} + 2000\left(1+\frac{0{,}095}{12}\right)^{108} + 2500\left(1+\frac{0{,}095}{12}\right)^{36}$

$= R12\,676{,}77$

3.2. $12\,676{,}77 = 3000(1+i)^{12}$

$\therefore i = \sqrt[12]{\frac{12676{,}77}{3000}} - 1$

$\therefore i = 12{,}76\%$

Question 4

4.1. $A = 950\,000(1+0{,}09)^{5}$

$= R1\,461\,692{,}76$

4.2. $473\,499{,}81 = 950\,000(1-i)^{5}$

$\therefore -i = \sqrt[5]{\frac{473\,499{,}81}{950\,000}} - 1$

$\therefore i = 13\%$

4.3. $F_v = 1\,461\,692{,}76 - 473\,499{,}81 - 9500\left(1+\frac{0{,}11}{12}\right)^{24}$

$= 976\,367{,}08$

$\therefore 976\,367{,}08 = x\left[\frac{\left(1+\frac{0{,}11}{12}\right)^{61} - 1}{\frac{0{,}11}{12}}\right]$

$\therefore x = R12\,017{,}27$

Question 5

5.1. $$2400\,000 = x\left[\frac{1-\left(1+\frac{0,135}{12}\right)^{-180}}{\frac{0,135}{12}}\right]$$

$\therefore x = R\,31\,159,65$

5.2. $$B.O. = 2400\,000\left(1+\frac{0,135}{12}\right)^{n} - 32000\left[\frac{\left(1+\frac{0,135}{12}\right)^{n}-1}{\frac{0,135}{12}}\right]$$

5.3.1. $$2400\,000 = 28000\left[\frac{1-\left(1+\frac{0,135}{12}\right)^{-n}}{\frac{0,135}{12}}\right]$$

$$-\left(1+\frac{0,135}{12}\right)^{-n} = \frac{2400\,000}{28000} \times \frac{0,135}{12} - 1$$

$$\therefore -n = \log_{\left(1+\frac{0,135}{12}\right)}\left(-\frac{2400000}{28000}\times\frac{0,135}{12}+1\right)$$

$\therefore n = 297,8589\ldots$

$\therefore$ 298 months

5.3.2. $$y = \left(2400\,000 - 28000\left[\frac{1-\left(1+\frac{0,135}{12}\right)^{-297}}{\frac{0,135}{12}}\right]\right)\left(1+\frac{0,135}{12}\right)^{298}$$

$\therefore y = R\,24\,069,61$

Question 6

$$35684,90\left[\frac{1-\left(1+\frac{0,15}{12}\right)^{-36}}{\frac{0,15}{12}}\right]\left(1+\frac{0,15}{12}\right)^{4} = x\left[\frac{1-\left(1+\frac{0,15}{12}\right)^{-32}}{\frac{0,15}{12}}\right]$$

$\therefore x = R\,41\,227,25$